# Asymptotic Structure

Aim: to skate over the theory (background to Friedrich, Bray...)

The question: what is an isolated body in GR?

The answer: One with a metric which is asymptotically flat
(or de Sitter or anti-de Sitter)

which means "the same at infinity".

The idea: make infinity a finite boundary by rescaling metric

then "fields at large distances" becomes "fields on the boundary"
(a separation of fields from arena)

So: what is infinity like for de Sitter, anti-de Sitter, Minkowski?

Begin with product metric on $\mathbb{R} \times S^3$: $g = dT^2 - dR^2 - \sin^2 R\, d\omega^2$

($d\omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$)

$$\boxed{R_{\alpha\beta} = \lambda g_{\alpha\beta} \;:\; C_{\alpha\beta\gamma\delta} = 0 \;:\; 3 \text{ cases}}$$

- $\lambda = -3$ : de Sitter

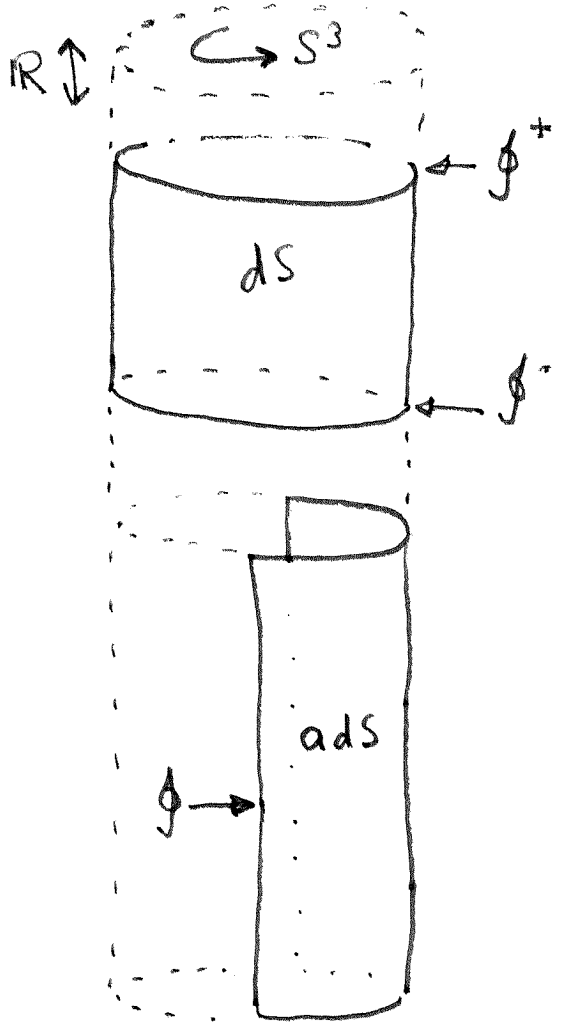
$$g = \Omega^2 \left[ dt^2 - \cosh^2 t \left( dr^2 + r^2 d\omega^2 \right) \right]$$

$$\Omega = \operatorname{sech} t = \sin T \;;\; r = R$$

- $\lambda = 3$ : anti-de Sitter

$$g = \Omega^2 \left[ \cosh^2 r\, dt^2 - dr^2 - \sinh^2 r\, d\omega^2 \right]$$

$$\Omega = \operatorname{sech} r = \cos R \;;\; t = T$$



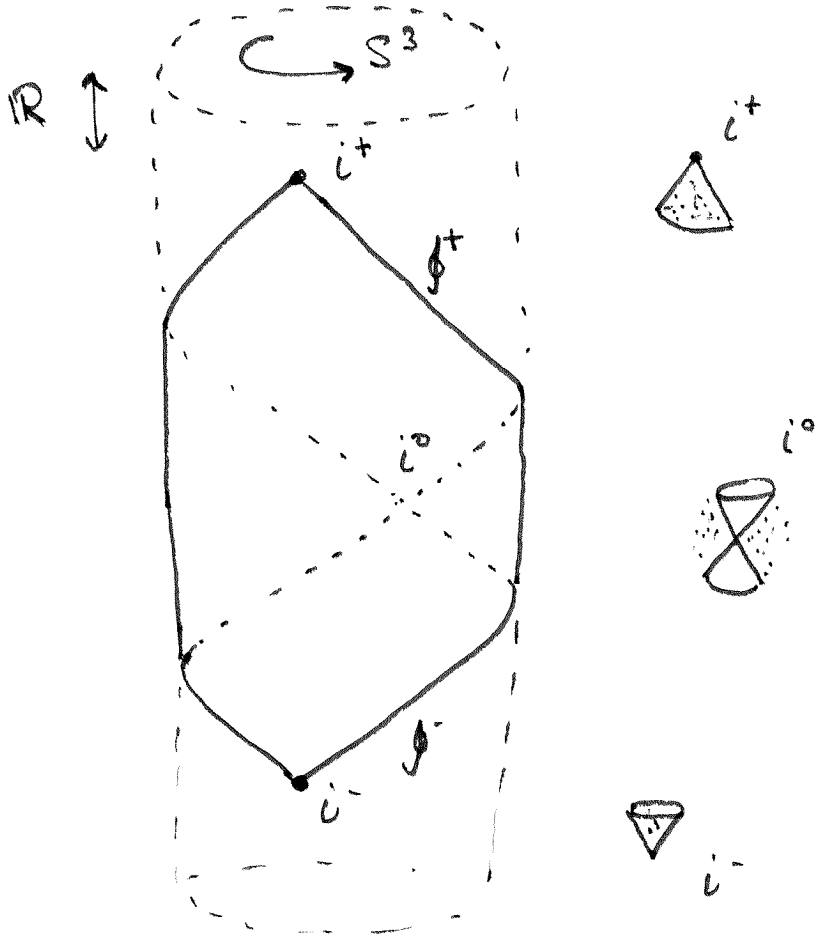
- $\lambda = 0$ : Minkowski

$$g = \Omega^2 [dt^2 - dr^2 - r^2 d\omega^2]$$

$$\Omega = \cos T + \cos R$$

$$= 2 \cos \tfrac{1}{2}(T+R) \cos \tfrac{1}{2}(T-R)$$

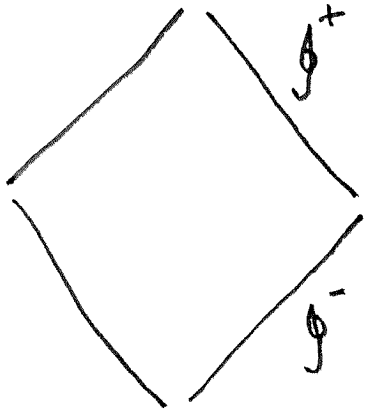
$$\sin T = 2\Omega t \; ; \; \sin R = 2\Omega r$$




c.f. Schwarzschild:

$$\Omega^2 \left[ \left(1 - \frac{2M}{r}\right) dW^2 \pm 2 dW dr - r^2 d\omega^2 \right]$$

$$\Omega = r^{-1}$$



So define: a smooth space-time $(\tilde{M}, \tilde{g})$ is asymptotically simple if $\exists$ another smooth $(M, g)$ and a smooth function $\Omega$ on $M$ such that

(i) $M$ is a manifold with boundary $\mathcal{I}$;

(ii) $\Omega > 0$ on $M \setminus \mathcal{I}$; $\Omega = 0$, $d\Omega \neq 0$ on $\mathcal{I}$

(iii) $\tilde{M} \cong M \setminus \mathcal{I}$ and $g = \Omega^2 \tilde{g}$ on $\tilde{M}$

(iv) every null geodesic in $\tilde{M}$ has 2 endpoints on $\mathcal{I}$.

- $(\tilde{M}, \tilde{g})$ is weakly asymptotically simple if its isometric to an asymptotically simple $(\tilde{M}', \tilde{g}')$ near to $\mathcal{I}$.
- define $k$-asymptotically simple if $\Omega$ is $C^k$ (and $M$ is $C^{k+1}$)
- are there any? many? uniqueness?

## Rescaling formulae:

physical ↓

$$g_{\alpha\beta} = \Omega^2 \tilde{g}_{\alpha\beta} \quad ; \quad g^{\alpha\beta} = \Omega^{-2} \tilde{g}^{\alpha\beta} \tag{1}$$

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - 2\Omega^{-1} \nabla_\alpha \nabla_\beta \Omega - g_{\alpha\beta} (\Omega^{-1} \Box\Omega - 3\Omega^{-2} |\nabla\Omega|^2) \tag{2}$$

$C^2$

$$\Omega^2 R = \tilde{R} - 6\Omega \Box\Omega + 12 |\nabla\Omega|^2 \tag{3}$$

Set $n_\alpha = \nabla_\alpha \Omega$ = normal to $\mathcal{I}$ and suppose $\tilde{R}_{\alpha\beta} = \lambda \tilde{g}_{\alpha\beta}$ near $\mathcal{I}$

(2) $\Rightarrow$ $$2\nabla_\alpha n_\beta + 2\Omega L_{\alpha\beta} - f g_{\alpha\beta} = 0 \tag{4}$$

$$L_{\alpha\beta} = \tfrac{1}{2}(R_{\alpha\beta} - \tfrac{1}{6} R\, g_{\alpha\beta}) \; ; \; f = \Omega^{-1}(n^\alpha n_\alpha + \tfrac{\lambda}{3}) \tag{5}$$

(3) $\Rightarrow$ $$\Omega R + 6 \Box\Omega = f \tag{6}$$

### First deductions

(6) $\Rightarrow$ $f$ continuous at $\mathcal{I}$, (5) $\Rightarrow$ $\mathcal{I}$ is timelike, spacelike, null as $\lambda<0, \lambda>0, \lambda=0$

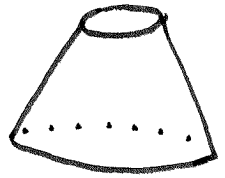
$C^2$

(4) $\Rightarrow$ $\mathcal{I}$ is umbilic ($\lambda \neq 0$) or shear-free ($\lambda = 0$)

Topology of $\mathscr{I}$ ($\lambda=0$): asymptotic simplicity $\Rightarrow \exists$ Cauchy surface $\Sigma$

Consider space of null geodesics $N \cong \Sigma \times S^2 \cong \mathscr{I}^+ \times \mathbb{R}^2$

$\therefore \mathscr{I}^+ \cong \mathbb{R} \times S^2$ ($\cong \mathscr{I}^-$) and $\Sigma \cong \mathbb{R}^3$



Gauge freedom: make another choice $\hat{g}_{\alpha\beta} = \hat{\Omega}^2 \tilde{g}_{\alpha\beta}$ so $\hat{\Omega} = \theta\Omega$

then at $\mathscr{I}$, $\hat{n}_\alpha = \theta n_\alpha$, $\hat{n}^\alpha = \theta^{-1} n^\alpha$, $\hat{h}_{\alpha\beta} = \theta^2 h_{\alpha\beta}$

$\uparrow g_{\alpha\beta}$ pulled back to $\mathscr{I}$

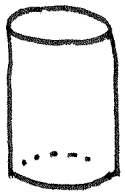
so $\Gamma_{\alpha\beta}{}^{\gamma\delta} = h_{\alpha\beta} n^\gamma n^\delta$ is gauge-invt.

Recall (6): $f = \Omega R + 6 \Box \Omega$, then $\theta^2 \hat{f} = 2 n^\alpha \theta_{,\alpha} + f$

gauge choice: for $\lambda = 0$ pick $f = 0$ then (4) $\Rightarrow \nabla_\alpha n_\beta = 0$ at $\mathscr{I}^\pm$

$\mathscr{I}$ is now expansion-free as well as shear-free

Residual gauge: $\theta$ with $n^\alpha \theta_{,\alpha} = 0$



Go back to (4): $2\nabla_\alpha n_\beta + 2\Omega L_{\alpha\beta} - f g_{\alpha\beta} = 0$ and differentiate. (use (5) & Riem)

$C^3$ $$n^\delta C_{\delta\gamma\alpha\beta} = 2\Omega \nabla_{[\alpha} L_{\beta]\gamma} \qquad (7)$$

**Second deduction** $$n^\delta C_{\delta\gamma\alpha\beta} = 0 \quad \underline{\text{at}} \quad \mathcal{I} \qquad (8)$$

So for $\lambda \neq 0$, $C_{\delta\gamma\alpha\beta} = 0$ at $\mathcal{I}$ but for $\lambda = 0$, 2 components remain

Differentiate again and use Bianchi identity:

$C^4$ $$(n_\delta \nabla_\lambda - n_\lambda \nabla_\delta) C^\delta{}_{\gamma\alpha\beta} = 2\Omega \nabla_\lambda \nabla_{[\alpha} L_{\beta]\gamma} = 0 \text{ at } \mathcal{I} \qquad (9)$$

and this gives an equation on $\mathcal{I}^+ \cong S^2 \times \mathbb{R}$ which forces $C_{\alpha\beta\gamma\delta} = 0$ at $\mathcal{I}$

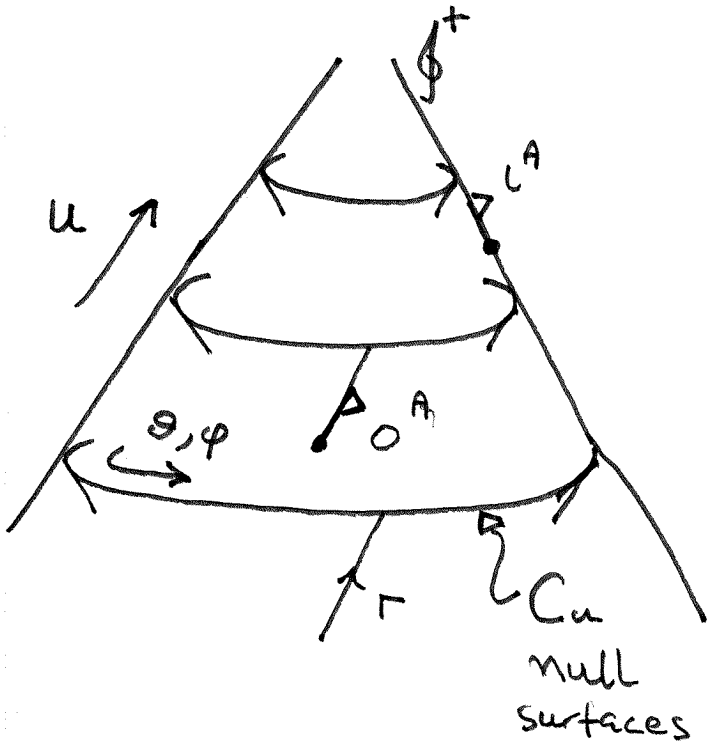
This fact finds expression in the Peeling Theorem.

Peeling Theorem : for this we need spinors so

$$T_{\mathbb{C}}M = S \otimes \bar{S} \quad ; \quad V^{\alpha} = V^{AA'} \quad ; \quad g_{\alpha\beta} = \epsilon_{AB}\,\epsilon_{A'B'}$$

$$C_{\alpha\beta\gamma\delta} = \Psi_{ABCD}\,\epsilon_{A'B'}\,\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\,\epsilon_{AB}\,\epsilon_{CD}$$

↑ Weyl spinor; symmetric; 5 cpx comp.




$r$ affine in $\tilde{M}$

$r\Omega \to 1$ at $\mathscr{I}^+$

normalised dyad $(o^A, l^A)$ ; $\epsilon_{AB} o^A l^B = 1$

Rescaling : $\epsilon_{AB} = \Omega\,\tilde{\epsilon}_{AB}$

$$\Psi_{ABCD} = \tilde{\Psi}_{ABCD}$$

and take components in the dyad.

$$\psi_0 = \psi_{ABCD}\, o^A o^B o^C o^D \;;\quad \psi_1 = \psi_{ABCD}\, o^A o^B o^C \iota^D$$

$$\psi_2 = \psi_{ABCD}\, o^A o^B \iota^C \iota^D \;;\quad \psi_3 = \psi_{ABCD}\, o^A \iota^B \iota^C \iota^D \;;\quad \psi_4 = \psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D$$

Now $l_A \bar{l}_{A'} = n_{AA'} = -\nabla_a \Omega \sim \Omega^2 \nabla_a r = \Omega^2 \tilde{l}_A \bar{\tilde{l}}_{A'}$

so choose $l_A = \Omega \tilde{l}_A \;;\; o_A = \tilde{o}_A \;;\; \iota^A = \tilde{\iota}^A \;;\; o^A = \Omega^{-1} \tilde{o}^A$

$C^3$ (8) $\Rightarrow \psi_{ABCD} l^D = 0$ at $\mathscr{I} \Rightarrow \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0$ at $\mathscr{I}$

$C^4$ (9) $\Rightarrow l^A \nabla_{AA'} \psi_{BCDE} = 0$ —"— $\Rightarrow \psi_0 = 0$ at $\mathscr{I}$ ; $\Omega^{-1}\psi_{ABCD}$ cont. at $\mathscr{I}$

so $\tilde{\psi}_4 = \psi_4 = r^{-1}\psi_4^0 + o(r^{-1})$

$\tilde{\psi}_3 = \Omega\psi_3 = r^{-2}\psi_3^0 + o(r^{-2})$

$\tilde{\psi}_2 = \Omega^2\psi_2 = r^{-3}\psi_2^0 + o(r^{-3})$

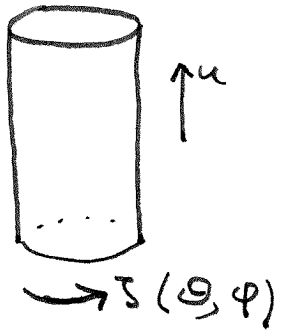
$\tilde{\psi}_1 = \Omega^3\psi_1 = r^{-4}\psi_1^0 + o(r^{-4})$

$\tilde{\psi}_0 = \Omega^4\psi_0 = r^{-5}\psi_0^0 + o(r^{-5})$

Peeling!

Departures from this signal lower differentiability at $\mathscr{I}^+$.

# Universal structure at $\mathscr{I}^+$ : ($\lambda = 0$) (Arena)




Set $f = 0$ ; use residual freedom to set

$$h = -\frac{4\, d\zeta\, d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} \qquad \text{unit sphere}$$

(Bondi coords)

Choose $u$ by $n^\alpha u_{,\alpha} = 1$

Symmetries : $\zeta \to \hat{\zeta} = \dfrac{a\zeta + b}{c\zeta + d}$ $\qquad SL(2,\mathbb{C}) \equiv O_+^\uparrow(1,3)$

$$h \to \hat{h} = V^{-2} h \quad , \quad V = (1+\zeta\bar{\zeta})^{-1}(|a\zeta+b|^2 + |c\zeta+d|^2)$$

$$u \to \hat{u} = V^{-1}(u + \alpha(\zeta,\bar{\zeta}))$$

↑ "supertranslation"

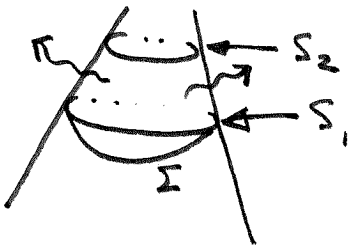
**the BMS group**

e.g. $\alpha = \alpha_0 + \alpha_1 \sin\theta\cos\varphi + \alpha_2 \sin\theta\sin\varphi + \alpha_3 \cos\theta$

↑ translation

$\exists!$ 4p normal subgroup

Fields at $\mathcal{I}$ : $\nabla_\alpha$ ; $\psi_i^o$ ; matter fields




Momentum

$$P_\mu = \int_S (\quad)(1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\, dS$$

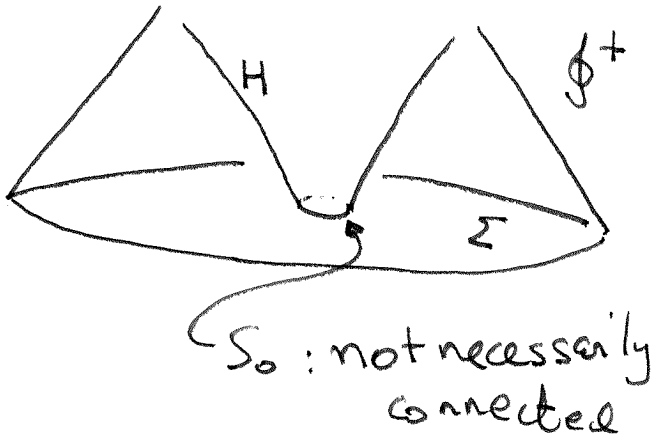
- $P[S_1] - P[S_2]$ is time-like, future pointing "Mass-loss" with radiation.
- $P[S_1]$ is time-like (given conditions on $\Sigma$) "Positive mass" D.E.C.

Angular Momentum ?

$$M_{\mu\nu} = \int_S (\quad)\, m_{\mu\nu}(\theta, \varphi)\, dS$$

agreement for stationary, not with radiation; origins?

# The Penrose Inequality




vacuum, $H = \partial(\text{past of } \mathscr{I}^+) = \text{horizon}$

$m_0$ = ADM mass at $\Sigma$

$m_\infty$ = final Bondi mass

$A_0$ = area of $S^0$

$A_\infty$ = final area of section of $H$

mass-loss $\Rightarrow$ $m_0 \geq m_\infty$

horizon area thm $\Rightarrow$ $A_\infty \geq A_0$

black hole uniqueness $\Rightarrow$ $16\pi m_\infty^2 \geq A_\infty$ (assuming "settles")

<u>so</u> $16\pi m_0^2 \geq 16\pi m_\infty^2 \geq A_\infty \geq A_0$

i.e. $16\pi m_0^2 \geq A_0$

## Existence theorems: are there solutions with $\mathscr{I}$?

- Christodoulou - Klainerman : $\lambda = 0$

  small, strongly asymp. flat data → Minkowski along any geodesic

  $\Psi_2, \Psi_3, \Psi_4$ peel but <u>not</u> $\Psi_0, \Psi_1$. ($\mathscr{I}$ not $C^3$)

- Friedrich : $\lambda < 0$

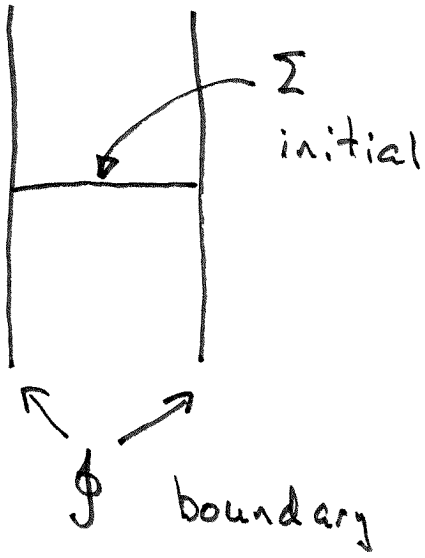
  data close to de Sitter ⇒ asymptotically simple

  data on $\mathscr{I}^-$ ⇒ asymptotically simple

- Friedrich : $\lambda > 0$

  ∃ a boundary-initial-value problem

  for asymptotically simple



- Friedrich: $\Lambda = 0$ / smooth hyperboloidal data $\Rightarrow$ asymp. simple

  small $\Rightarrow \exists\, i^+$

  A.C.F. $\exists$ smooth hyperboloidal data

  & $\exists$ phg hyperboloidal data

- using this $\Rightarrow$ $\exists$ Cauchy data with smooth $\mathscr{I}^+$ (C.D.; K.N.) (C.W. too)

- The Big Question: <u>which</u> Cauchy data evolve to give a smooth $\mathscr{I}$ ?