Asymptotic Structure

Aim: to skate over the theory (background to Friedrich, Bray...)

The question: what is an isolated body in GR?

The answer: One with a metric which is asymptotically flat (or de Sitter or anti-de Sitter)

which means "the same at infinity".

The idea: make infinity a finite boundary by rescaling metric

then "fields at large distances" becomes "fields on the boundary"

(a separation of fields from arena)

So: what is infinity like for de Sitter, anti de Sitter, Minkowski?

Begin with product metric on $\mathbb{R} \times S^3$: $g = dT^2 - dR^2 - \sin^2 R\, d\omega^2$

$d\theta^2 + \sin^2\theta\, d\phi^2$

$$\boxed{R_{\alpha\beta} = \lambda g_{\alpha\beta} \;:\; C_{\alpha\beta\gamma\delta} = 0 : 3 \text{ cases}}$$

- $\lambda = -3$: de Sitter

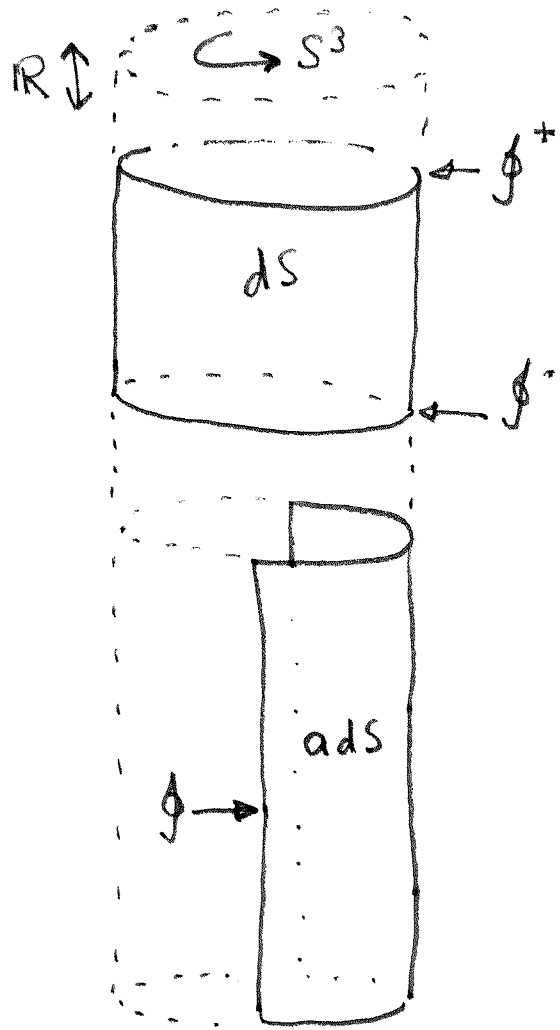
$$g = \Omega^2 [dt^2 - \cosh^2 t\,(dr^2 + r^2 d\omega^2)]$$

$$\Omega = \operatorname{sech} t = \sin T \;;\; r = R$$

- $\lambda = 3$: anti-de Sitter

$$g = \Omega^2 [\cosh^2 r\, dt^2 - dr^2 - \sinh^2 r\, d\omega^2]$$

$$\Omega = \operatorname{sech} r = \cos R \;;\; t = T$$

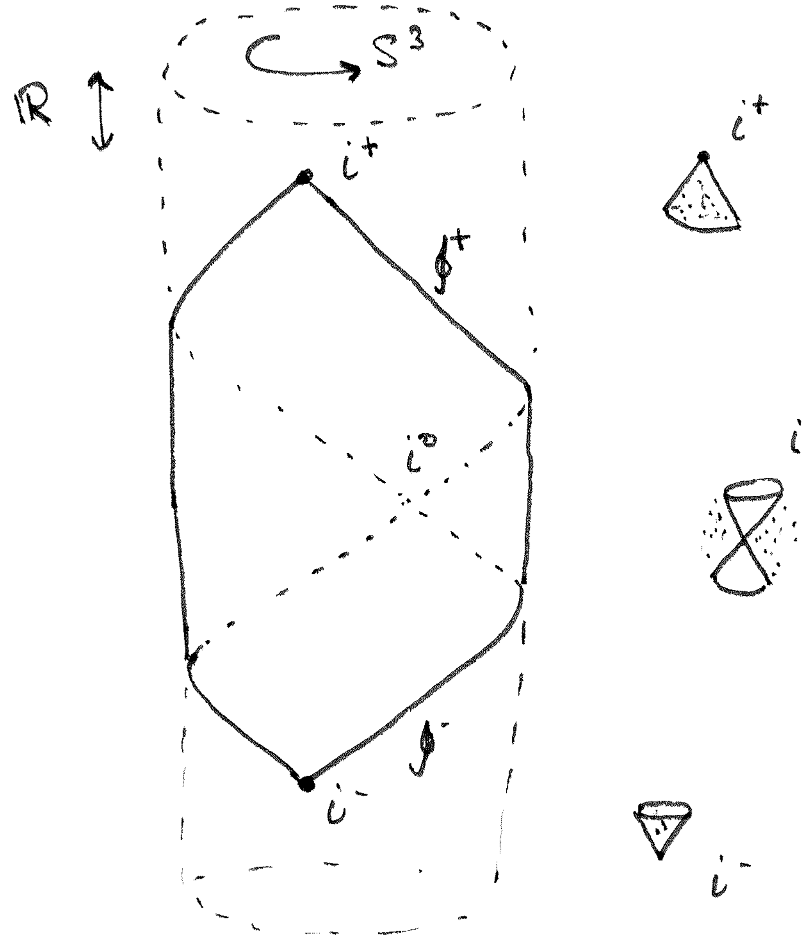


- $\lambda = 0$: Minkowski

$$g = \Omega^2 [dt^2 - dr^2 - r^2 d\omega^2]$$

$$\Omega = \cos T + \cos R$$
$$= 2 \cos \tfrac{1}{2}(T+R) \cos \tfrac{1}{2}(T-R)$$

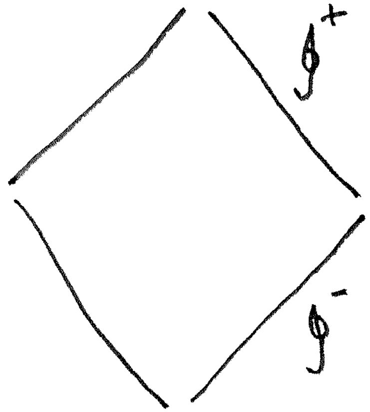
$$\sin T = 2\Omega t \;;\; \sin R = 2\Omega r$$



cf. Schwarzschild:

$$\Omega^2 \left[\left(1 - \frac{2M}{r}\right) dW^2 \pm 2\, dW\, dr - r^2 d\omega^2 \right]$$

$$\Omega = r^{-1}$$



So define: a smooth space-time $(\tilde{M}, \tilde{g})$ is *asymptotically simple* if $\exists$ another smooth (M, g) and a smooth function Ω on M such that

(i) M is a manifold with boundary $\mathscr{I}$;

(ii) $\Omega > 0$ on $M \setminus \mathscr{I}$; $\Omega = 0$, $d\Omega \neq 0$ on $\mathscr{I}$

(iii) $\tilde{M} \cong M \setminus \mathscr{I}$ and $g = \Omega^2 \tilde{g}$ on $\tilde{M}$

(iv) every null geodesic in $\tilde{M}$ has 2 endpoints on $\mathscr{I}$.

- $(\tilde{M}, \tilde{g})$ is *weakly* asymptotically simple if its isometric to an asymptotically simple $(\tilde{M}', \tilde{g}')$ near to $\mathscr{I}$.
- define k-asymptotically simple if Ω is C^k (and M is C^{k+1})
- are there any? many? uniqueness?

Rescaling formulae: physical

$$g_{\alpha\beta} = \Omega^2 \tilde{g}_{\alpha\beta} \quad ; \quad g^{\alpha\beta} = \Omega^{-2} \tilde{g}^{\alpha\beta} \qquad (1)$$

$$R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - 2\Omega^{-1} \nabla_\alpha \nabla_\beta \Omega - g_{\alpha\beta}\left(\Omega^{-1} \Box \Omega - 3\Omega^{-2} |\nabla\Omega|^2\right) \qquad (2)$$

C^2

$$\Omega^2 R = \tilde{R} - 6\Omega \Box \Omega + 12 |\nabla \Omega|^2 \qquad (3)$$

Set $n_\alpha = \nabla_\alpha \Omega$ = normal to $\mathscr{I}$ and suppose $\tilde{R}_{\alpha\beta} = \lambda \tilde{g}_{\alpha\beta}$ near $\mathscr{I}$

$$(2) \Rightarrow \quad 2\nabla_\alpha n_\beta + 2\Omega L_{\alpha\beta} - f g_{\alpha\beta} = 0 \qquad (4)$$

$$L_{\alpha\beta} = \tfrac{1}{2}\left(R_{\alpha\beta} - \tfrac{1}{6} R g_{\alpha\beta}\right) ; \; f = \Omega^{-1}\left(n^\alpha n_\alpha + \tfrac{\lambda}{3}\right) \qquad (5)$$

$$(3) \Rightarrow \quad \Omega R + 6 \Box \Omega = f \qquad (6)$$

First deductions

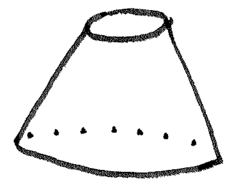
C^2

(6) $\Rightarrow$ f continuous at $\mathscr{I}$, (5) $\Rightarrow$ $\mathscr{I}$ is timelike, spacelike, null as $\lambda < 0, \lambda > 0, \lambda = 0$

(4) $\Rightarrow$ $\mathscr{I}$ is umbilic ($\lambda \neq 0$) or shear-free ($\lambda = 0$)

<u>Topology of $\mathscr{I}$</u> ($\lambda = 0$): asymptotic simplicity $\Rightarrow \exists$ Cauchy surface Σ

Consider space of null geodesics $N \cong \Sigma \times S^2 \cong \mathscr{I}^+ \times \mathbb{R}^2$



$\therefore\ \mathscr{I}^+ \cong \mathbb{R} \times S^2$ $(\cong \mathscr{I}^-)$ and $\Sigma \cong \mathbb{R}^3$

<u>Gauge freedom</u>:

make another choice $\hat{g}_{\alpha\beta} = \hat{\Omega}^2 \tilde{g}_{\alpha\beta}$ so $\hat{\Omega} = \theta\Omega$

then at $\mathscr{I}$, $\hat{n}_\alpha = \theta n_\alpha$, $\hat{n}^\alpha = \theta^{-1} n^\alpha$, $\hat{h}_{\alpha\beta} = \theta^2 h_{\alpha\beta}$

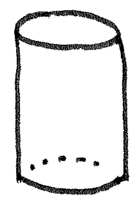
↑ $g_{\alpha\beta}$ pulled back to $\mathscr{I}$

so $\Gamma_{\alpha\beta}{}^{\gamma\delta} = h_{\alpha\beta} n^\gamma n^\delta$ is gauge-invt.

Recall (6): $f = \Omega R + 6 \Box \Omega$, then $\theta^2 \hat{f} = 2 n^\alpha \theta_{,\alpha} + f$

gauge choice: for $\lambda = 0$ pick $f = 0$ then (4) $\Rightarrow \nabla_\alpha n_\beta = 0$ at $\mathscr{I}^\pm$

$\mathscr{I}$ is now expansion-free as well as shear-free



<u>Residual gauge</u>: θ with $n^\alpha \theta_{,\alpha} = 0$

Go back to (4): $2\nabla_\alpha n_\beta + 2\Omega L_{\alpha\beta} - f g_{\alpha\beta} = 0$ and differentiate: (use (5) & Riem)

C^3

$$n^\delta C_{\delta\gamma\alpha\beta} = 2\Omega \nabla_{[\alpha} L_{\beta]\gamma} \qquad (7)$$

Second deduction $\quad n^\delta C_{\delta\gamma\alpha\beta} = 0$ at $\mathscr{I}$ (8)

So for $\lambda \neq 0$, $C_{\delta\gamma\alpha\beta} = 0$ at $\mathscr{I}$ but for $\lambda = 0$, 2 components remain

Differentiate again and use Bianchi identity:

C^4

$$(n_\delta \nabla_\lambda - n_\lambda \nabla_\delta) C^\delta{}_{\gamma\alpha\beta} = 2\Omega \nabla_\lambda \nabla_{[\alpha} L_{\beta]\gamma} = 0 \text{ at } \mathscr{I} \qquad (9)$$

and this gives an equation on $\mathscr{I}^+ \cong S^2 \times \mathbb{R}$ which forces $C_{\alpha\beta\gamma\delta} = 0$ at $\mathscr{I}$

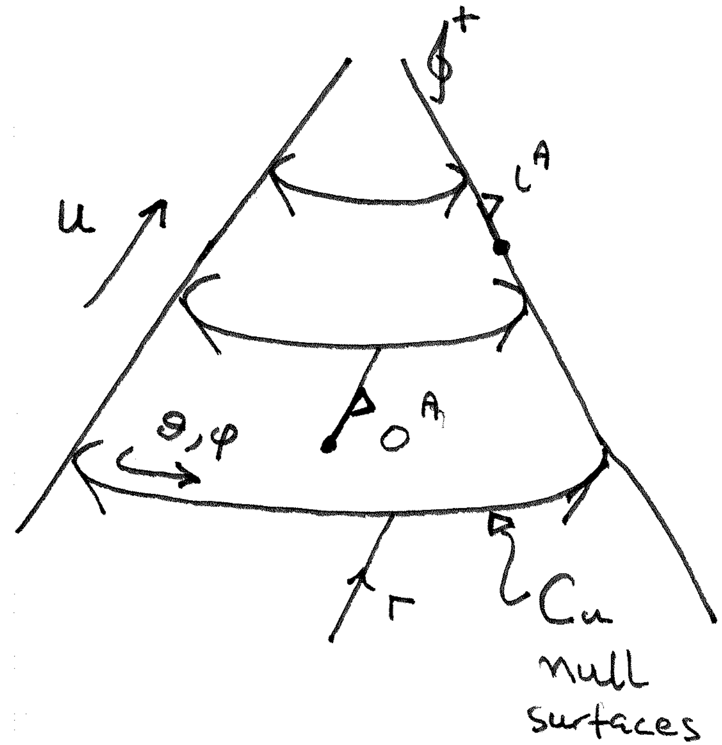
This fact finds expression in the Peeling Theorem.

Peeling Theorem: for this we need spinors so

$$T_{\mathbb{C}} M = S \otimes \bar{S} \quad ; \quad V^{\alpha} = V^{AA'} \quad ; \quad g_{\alpha\beta} = \epsilon_{AB}\,\epsilon_{A'B'}$$

$$C_{\alpha\beta\gamma\delta} = \Psi_{ABCD}\,\epsilon_{A'B'}\,\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\,\epsilon_{AB}\,\epsilon_{CD}$$

Weyl spinor; symmetric; 5 cpx comp.



r affine in $\tilde{M}$

$r\Omega \to 1$ at $\mathscr{I}^+$

normalised dyad (o^A, ι^A) ; $\epsilon_{AB}\, o^A \iota^B = 1$

Rescaling: $\epsilon_{AB} = \Omega\, \tilde{\epsilon}_{AB}$

$$\Psi_{ABCD} = \tilde{\Psi}_{ABCD}$$

and take components in the dyad.

$$\Psi_0 = \Psi_{ABCD}\, o^A o^B o^C o^D \;;\quad \Psi_1 = \Psi_{ABCD}\, o^A o^B o^C \iota^D$$

$$\Psi_2 = \Psi_{ABCD}\, o^A o^B \iota^C \iota^D \;;\quad \Psi_3 = \Psi_{ABCD}\, o^A \iota^B \iota^C \iota^D \;;\quad \Psi_4 = \Psi_{ABCD}\, \iota^A \iota^B \iota^C \iota^D$$

Now $\iota_A \bar{\iota}_{A'} = n_{AA'} = -\nabla_\alpha \Omega \sim \Omega^2 \nabla_\alpha r = \Omega^2 \tilde{\iota}_A \tilde{\bar{\iota}}_{A'}$

so choose $\iota_A = \Omega \tilde{\iota}_A \;;\; o_A = \tilde{o}_A \;;\; \iota^A = \tilde{\iota}^A \;;\; o^A = \Omega^{-1} \tilde{o}^A$

C^3 (8) $\Rightarrow \Psi_{ABCD}\, \iota^D = 0$ at $\mathscr{I}$ $\Rightarrow \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$ at $\mathscr{I}$

C^4 (9) $\Rightarrow \iota^A \nabla_{AA'} \Psi_{BCDE} = 0$ —"— $\Rightarrow \Psi_0 = 0$ at $\mathscr{I}$; $\Omega^{-1}\Psi_{ABCD}$ cont. at $\mathscr{I}$

so $\tilde{\Psi}_4 = \Psi_4 = r^{-1} \overset{\circ}{\Psi}_4 + o(r^{-1})$

$\tilde{\Psi}_3 = \Omega \Psi_3 = r^{-2} \overset{\circ}{\Psi}_3 + o(r^{-2})$

$\tilde{\Psi}_2 = \Omega^2 \Psi_2 = r^{-3} \overset{\circ}{\Psi}_2 + o(r^{-3})$

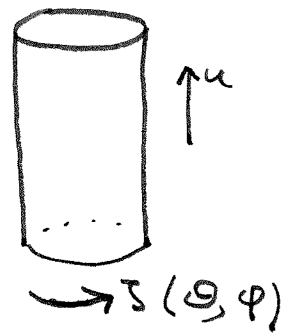
$\tilde{\Psi}_1 = \Omega^3 \Psi_1 = r^{-4} \overset{\circ}{\Psi}_1 + o(r^{-4})$

$\tilde{\Psi}_0 = \Omega^4 \Psi_0 = r^{-5} \overset{\circ}{\Psi}_0 + o(r^{-5})$

Peeling!

Departures from this signal lower differentiability at $\mathscr{I}^+$.

Universal structure at $\mathscr{I}^+$: $(\Lambda = 0)$ (Arena)



Set $f = 0$; use residual freedom to set

$$h = -\frac{4\, d\zeta\, d\bar\zeta}{(1+\zeta\bar\zeta)^2}$$ unit sphere

(Bondi coords)

choose u by $n^\alpha u_{,\alpha} = 1$

symmetries : $\zeta \to \hat\zeta = \dfrac{a\zeta + b}{c\zeta + d}$ $\quad SL(2,\mathbb{C}) \cong O_+^{\uparrow}(1,3)$

$h \to \hat h = V^{-2} h$, $V = (1+\zeta\bar\zeta)^{-1}(|a\zeta+b|^2 + |c\zeta+d|^2)$

$u \to \hat u = V^{-1}(u + \alpha(\zeta,\bar\zeta))$

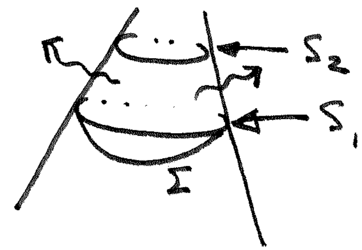
↑ "supertranslation"

$\boxed{\text{the BMS group}}$

e.g. $\alpha = \alpha_0 + \alpha_1 \sin\theta\cos\varphi + \alpha_2 \sin\theta \sin\varphi + \alpha_3 \cos\theta$

↑ translation

$\exists!$ 4P normal subgroup



Fields at $\mathscr{I}$: ∇_α ; ψ_i^0 ; matter fields

Momentum

$$P_{\underline{\mu}} = \int_S (\qquad)(1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\, dS$$

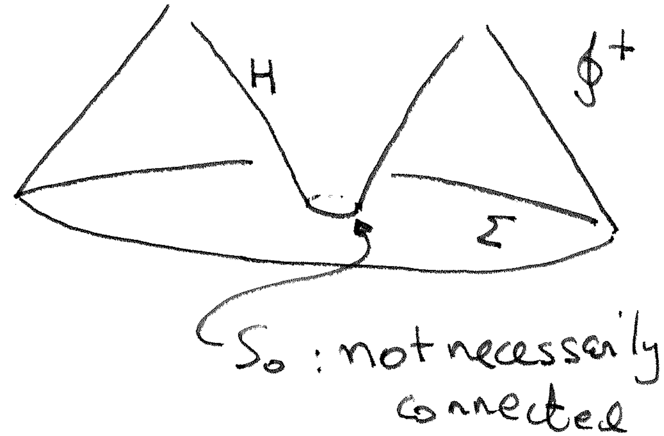
- $P[S_1] - P[S_2]$ is time-like, future pointing "Mass-loss" with radiation.
- $P[S_1]$ is time-like (given conditions on Σ) D.E.C. "Positive mass"

Angular Momentum ?

$$M_{\underline{\mu\nu}} = \int_S (\qquad)\, m_{\underline{\mu\nu}}(\theta, \varphi)\, dS$$

agreement for stationary, not with radiation; origins?

The Penrose Inequality



vacuum, $H = \partial(\text{past of } \mathscr{I}^+)$ = horizon

M_0 = ADM mass at Σ

m_∞ = final Bondi mass

A_0 = area of S^0

A_∞ = final area of section of H

mass-loss $\Rightarrow$ $m_0 \geq m_\infty$

horizon area thm $\Rightarrow$ $A_\infty \geq A_0$

black hole uniqueness $\Rightarrow$ $16\pi m_\infty^2 \geq A_\infty$ (assuming "settles")

so $$16\pi m_0^2 \geq 16\pi m_\infty^2 \geq A_\infty \geq A_0$$

i.e. $$16\pi m_0^2 \geq A_0$$

Existence theorems: are there solutions with $\mathscr{I}$?

- Christodoulou - Klainerman : $\lambda = 0$

small, strongly asymp. flat data $\rightarrow$ Minkowski along any geodesic

Ψ_2, Ψ_3, Ψ_4 peel but not Ψ_0, Ψ_1. ($\mathscr{I}$ not C^3)

- Friedrich : $\lambda < 0$

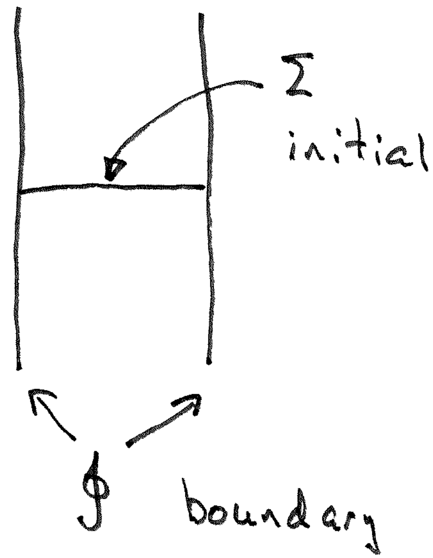
data close to de Sitter $\Rightarrow$ asympotically simple

data on $\mathscr{I}^-$ $\Rightarrow$ asymptotically simple

- Friedrich : $\lambda > 0$

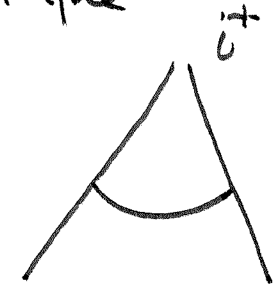
$\exists$ a boundary-initial-value problem

for asymptotically simple



- Friedrich : $\lambda = 0$ / smooth hyperboloidal data $\Rightarrow$ asymp. simple

  small $\Rightarrow$ $\exists\, i^+$



  A.C.F. $\exists$ smooth hyperboloidal data

  & $\exists$ phg hyperboloidal data

- using this $\mathscr{I}$ $\exists$ Cauchy data with smooth $\mathscr{I}^+$ (C.D.; K.N.)
  (C.W. too)

- The Big Question: <u>which</u> Cauchy data evolve

  to give a smooth $\mathscr{I}$?