Results on the asymptotic conformal structure of asymptotically flat vacuum solutions with vanishing cosmological constant.

Two types of initial value problems.

Most important:

standard Cauchy problem, data on a space-like hypersurface S with asymptotically euclidean end(s).

Standard example:

$\tilde{S} = \{t = 0\} \sim I\!R^3$ **in Minkowski space, induced data** $-\tilde{h} = \tilde{e}$, $\tilde{\chi} = 0$.

By embedding of Minkowski space into the Einstein cosmos:

$$\tilde{S} \to S = \tilde{S} \cup \{i\} \sim S^3, \quad \tilde{e} \to d\omega^2, \quad \tilde{\chi} \to \chi = 0,$$

with i representing space-like infinity for $(\tilde{S}, \tilde{e})$, $d\omega^2 = \Omega^2\, \tilde{e}$ **on** $\tilde{S}$,

$\Omega \in C^\infty(S)$ **with** $\Omega = 0$, $d\,\Omega = 0$, $Hess(\Omega) = c\, d\omega^2$ **at** i, $c \neq 0$.

If $\tilde{h}, \tilde{\chi} \in C^\infty(\tilde{S})$ **are asymptotically euclidean vacuum data, then** $h = \Omega^2\, \tilde{h}$ **is not necessarily in** $C^\infty(S)$. **But even if** $h \in C^\infty(S)$ **then**

$$d^\mu{}_{\nu\lambda\rho} = O(r^{-3}) \quad \textbf{as} \quad r \to 0 \quad \textbf{unless} \quad m_{ADM} = 0.$$

where r denotes the h-distance from i.

As a preparation study:

hyperboloidal initial value problem, data on a hypersurface H with boundary Σ, thought of as being embedded into an asymptotically simple space-time such that $\Sigma = H \cap \mathcal{J}^+$ and H is space-like. Σ may have several components.

Standard example:

extension of the space-like hyperboloid $\{t^2 - |x|^2 = 1, t > 0\}$ **in Minkowski space to $\mathcal{J}^+$ with induced data. Space of constant negative curvature.**

Basic differences:

Hyperboloidal problem intrinsically non-time-symmetric,

$$\Omega(p) \sim dist(p, i)^2 \ \textbf{ on } S \quad \textbf{while} \quad \Omega(p) \sim dist(p, \Sigma) \ \textbf{ on } H.$$

Hyperboloidal initial data.

Construction of hyperboloidal data with non-vanishing, constant, mean extrinsic physical curvature following the standard procedure:

L. Andersson, P.T. Chruściel, H. F., CMP, 1992
L. Andersson, P.T. Chruściel, CMP 1994, Diss. Math. 1996.

Prescribe certain 'free data': conformal metric h_{ab} and 'trial $\hat{\chi}_{ab}$' on H,

obtain 'remaining data' (Ω, χ_{ab}) by solving (degenerate) elliptic equations on H,

obtain 'complete conformal data' ($d^i{}_{jkl}, \ldots$) by differentiation and algebra (involves divisions by conformal factor).

Subtleties occur near Σ:

a) for $h_{ab}, \hat{\chi}_{ab} \in C^\infty(H)$ the remaining data and the complete conformal data $\Omega, \chi_{ab}, d^i{}_{jkl}, \ldots \in C^\infty(H \setminus \Sigma)$ have in general a 'polylogarithmic expansions' at Σ, i.e. an expansion in terms of $x^k \log^j x$, where x denotes the h-distance from Σ.
In particular: $d^i{}_{jkl}$ is in general unbounded near Σ.

b) if $h_{ab}, \hat{\chi}_{ab} \in C^\infty(H)$ satisfy (a finite number of) 'regularity conditions' at Σ the remaining data Ω, χ_{ab} as well as the complete conformal data $d^i{}_{jkl}, \ldots$ are smooth on H.

c) if $h_{ab}, \hat{\chi}_{ab} \in C^\infty(H \setminus \Sigma)$ have polylogarithmic expansions at Σ, then the complete conformal data $d^i{}_{jkl}, \ldots$ have polylogarithmic expansions at Σ.

What can be said about the evolution of these data ?

Evolution of hyperboloidal data.

The conformal field equations have been used to obtain the following results:

Case (b):

the data evolve into a solution which admits a smooth 'piece of $\mathcal{J}^+$' in the sense that it satisfies the first three conditions of asymptotic simplicity.

If the data are sufficiently close to Minkowskian hyperboloidal data the solution is null geodesically complete in the future and admits a smooth conformal extension with a regular point i^+ which represents future time-like infinity.

Note: it is a non-trivial consequence of the conformal field equations that they force the null generators of $\mathcal{J}^+$ in a suitable gauge to meet in the future at precisely one point.

Case (a) (Chruściel and Lengard (2001)):

the data evolve into a solution on a manifold $M \sim H \times [0,1[$ with boundary $\partial M \sim \Sigma \times [0,1[$ such that the solution is smooth on $M \setminus \partial M$,

∂M is 'null' in the sense that it is a limit of smooth null hypersurfaces in $M \setminus \partial M$,
$\Omega \to 0$, $d\Omega \to \neq 0$ at ∂M.

The behaviour of the solution near ∂M is controlled in terms of certain weighted Sobolev spaces which admit singularities of the form $d^i{}_{jkl} \sim \frac{1}{x}$ with x the coordinate distance from ∂M.

The solution is expected to admit a polylogarithmic expansion in terms of x.

Which of these solutions can arise by Einstein evolution from asymptotically flat standard Cauchy data ?

Existence of asymptotically simple solutions.

The difficulties at space-like infinity are avoided if the data evolve into a solution which is known explicitly near space-like infinity.

P. Chruściel, E. Delay, CQG, 2002 (cf. also J. Corvino, CMP, 2000):

Given smooth, asymptotically flat, time-symmetric initial data $(h^*_{ab}, I\!R^3)$ **satisfying the vacuum constraint** $R[h^*] = 0$ **and the reflection symmetry** $h^*_{ab}(x) = h^*_{ab}(-x)$, **then for given** $R > 0$, $k \geq k_* > 0$

(i) $\exists\ h_{ab} \in C^k(I\!R^3)$ **such** $h_{ab} = h^*_{ab}$ **for** $|x| < R$, $h_{ab} =$ **Schwarzschild**$_{m\geq 0}$ **for** $|x| > 2\,R$ **and** $R[h] = 0$ **on** $I\!R^3$.

(ii) $\exists\ C^0$**-families** $h_{ab}(\lambda)$, $\lambda \in [0, 1[$, **as above s.t.** $m_h > 0$ **for** $\lambda > 0$, **and** $h_{ab} \to \delta_{ab}$ **as** $\lambda \to 0$ **with fixed** R **and** k.

The time evolution of these data admit smooth hyperboloidal hypersurfaces for which a part coincides with the $|x| \leq 2\,R$ **part of the Cauchy hypersurface and the rest lies in the Schwarzschild part of the evolution.**

These hyperboloidal hypersurfaces/data can be constructed such that they approach Minkowskian hyperboloidal data as $m_h \to 0$.

The results on the hyperboloidal initial value problem imply:

There exist non-trivial asymptotically simple solutions which admit complete conformal extension (including regular points $i^{\pm}$**) of class** C^k **for specified** k.

. . . finally, after 40 years . . . . . .

Very special: $d^i{}_{jkl}(i^+) = 0$.

For small m_h: $\exists$ **global conformal Gauss systems.**

Possibility to calculate numerically entire space-times by solving the general conformal field equations ?

'$\exists$ **regular** $i^{\pm}$**' not an end in itself. What happens for large** m_h **?**

The problems at space-like infinity.

The standard compactification adds a point i at infinity to the space-like Cauchy hypersurface: $\tilde{S} \equiv \{t = 0\} \rightarrow S = \tilde{S} \cup \{i\}$

We have on S (r denoting a radial coordinate with $r(i) = 0$):
a conformal 3-metric $h = O(1)$, which can be chosen to be smooth,
a conformal factor $\Omega = O(r^2)$, which can not be smooth if $m \neq 0$,
a trace free conformal second fundamental form $\chi_{ab} = \Omega^2\,\psi_{ab}$, $\psi_{ab} = O(\frac{1}{r^4})$.

The rescaled conformal Weyl tensor then has the electric part

$$d_{ab} = \Omega^{-2}\left\{D_a\,D_b\Omega - \frac{1}{3}\,h_{ab}\,D^c\,D_c\Omega + \Omega\,s_{ab}\right\}$$

$$-\Omega^3\left\{\psi^c{}_c\,\psi_{ab} - \psi^c{}_a\,\psi_{cb} - \frac{1}{3}\,h_{ab}\,\left((\psi^c{}_c)^2 - \psi^{cd}\,\psi_{cd}\right)\right\} = O(\frac{1}{r^3}),$$

and the magnetic part

$$d^*_{ab} = -2\,D_c\,\Omega\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} - \Omega\,D_c\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} = O(\frac{1}{r^3}).$$

S. Dain, H. F., CMP 2001:
There exists a large class of data with h smooth at i and $\psi_{ab} = O(\frac{1}{r^3})$ such that all data for the conformal field equations have an expansion in r^k at i (i.e. no $\log r$-terms).

If $\psi_{ab} = O(\frac{1}{r^3})$: linear ADM-momentum $= 0$, ADM-angular momentum $\neq 0$ possible. Then $d_{ab} = O(\frac{1}{r^3})$, $d^*_{ab} = O(\frac{1}{r^2})$.
If $\chi_{ab} = 0$ then $d^*_{ab} = 0$ but $d_{ab} = O(\frac{1}{r^3})$, unless $m = 0$.

Consider in the following data on S with h_{ab} smooth and $\psi_{ab} = O(\frac{1}{r^3})$ near i such that all conformal data admit an expansion in r^k.

Nevertheless: the nature of the r-ϑ-relations (ϑ denoting the angular variables) destroys also the smoothness of terms of higher order in an expansion in terms of r^k.
Details of formal expansion type analysis quickly become complicated.

Gauge to be chosen smooth or rough at i ?

Transport of gauge by wave equations generates 'roughness' at $\mathcal{J}^{\pm}$?

Localization of $\mathcal{J}^{\pm} = \{\Omega = 0\}$ possible if Ω is 'rough' ?

A new differentiable structure at space-like infinity.

Define a setting which admits a convenient analysis of the r-ϑ-relations.

Choose a conformal scaling for the initial data near i and an oriented h-orthonormal frame $\{e_a\}_{a=1,2,3}$ at i.

Set $e_a(s) = s^c{}_a \, e_c$ with $s = (s^c{}_a) \in SO(3)$, transport $e_a(s)$ parallely along h-geodesics tangent to $e_3(s)$ at i.

Denote by ρ the affine parameter on the geodesics with $< e_3, \rho >= 1$, $\rho(i) = 0$ and denote by $e_a(\rho, s)$ the frame obtained from $e_a(s)$ at value ρ.

Assume $|\rho| < a$, with $B_a(i)$ a convex h-normal neighbourhood. Then:
$]-a, a[\times SO(3) \ni (\rho, s) \xrightarrow{\Phi} e_a(\rho, s) \in SO(S)$ defines a smooth embedding into bundle $SO(S)$ of oriented orthonormal frames over S.

The set $\hat{B} \equiv \Phi([0, a[\times SO(3))$ has boundary $I'^0 \equiv \{\rho = 0\} \simeq SO(3)$ and projection $\hat{B} \xrightarrow{\pi} B_a(i)$ which maps I'^0 onto i.

The action of $SO(2)$ implies a factorization $\hat{B}' \xrightarrow{\pi'} B'' = \hat{B}'/SO(2) \xrightarrow{\pi''} B_a(i)$ where π'', which maps $\pi'(I^0) \simeq S^2$ onto i, can be used to identify $B'' \setminus \pi'(I^0)$ with $B_a(i) \setminus \{i\}$.

By this identification the inner point i is replaced by a boundary diffeomorphic to the sphere S^2: $B_a(i) \rightarrow (B_a(i) \setminus \{i\}) \cup S^2$.

Instead:

Since we will work with a spin frame formalism it is much more natural and convenient to pull back $\hat{B}'$ to the bundle of normalized spin frames, work on 4-dimensional manifold $\hat{B} = [0, a[\times SU(2) \sim [0, a[\times S^3$. Then $\hat{B}$ carries action by $U(1)$, is 'coordinatized' by ρ and $s \in SU(2)$, and has boundary $I^0 = \{0\} \times S^3$ which projects onto $\{i\}$.

A function f on $\hat{B}$ which is invariant under $U(1)$ descends to a function f_* on $B_a(i)$. Since $f \in C^\infty(\hat{B})$ need not imply that $f_* \in C^\infty(B_a(i))$, there are functions on B_a which lift to smooth functions on $\hat{B}$ without being smooth at i.

We shall only consider functions which transform homogeneously under the action of $U(1)$ (have a well defined spin weight).

The regular finite initial value problem near space-like infinity.

To formulate near space-like infinity a useful initial value problem for the general conformal field equations we choose a function $\kappa = \rho\,\mu$ with $\mu \in C^\infty(\hat{B})$ and $\mu = 1$ on I^0 and define a new conformal scaling
$\Omega \to \Theta_* = \kappa^{-1}\,\Omega$, $h \to \kappa^{-2}\,h$, $\ldots$

In this scaling $\Theta = O(\rho)$, the metric coefficients diverge at I^0, but the frame coefficients remain/become smooth.

Further suitable data for the conformal Gauss system give

$$\Theta = \Theta_* \left(1 - (\frac{\tau}{f})^2\right), \qquad d_0 = -2\,\tau\,g, \qquad d_a$$

$$f, g, d_a \in C^\infty(\hat{B}), \quad 1 \leq f < \infty \;\; \textbf{on} \;\; \hat{B}, \;\; f = 1, \;\; g = 0 \;\; \textbf{on} \;\; I^0.$$

The electric and the magnetic parts of the rescaled conformal Weyl tensor transform into

$$d_{ab} = \kappa^3\,\Omega^{-2}\left\{D_a\,D_b\Omega - \frac{1}{3}\,h_{ab}\,D^c\,D_c\Omega + \Omega\,s_{ab}\right\}$$

$$-\kappa^3\,\Omega^3\left\{\psi^c{}_c\,\psi_{ab} - \psi^c{}_a\,\psi_{cb} - \frac{1}{3}\,h_{ab}\,((\psi^c{}_c)^2 - \psi^{cd}\,\psi_{cd})\right\} = O(1),$$

$$d^*_{ab} = -2\,\kappa^3\,D_c\,\Omega\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} - \kappa^3\,\Omega\,D_c\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} = O(1),$$

and lift to smooth functions on $\hat{B}$. In fact all unknowns in the general conformal field equations lift to functions on $\hat{B}$ which can be extended smoothly through I^0 into the region where $\rho \leq 0$.

The hyperbolic equations are to be solved on the 'physical manifold'

$$\tilde{M} = \{\rho > 0, \quad s \in SU(2), \quad -f(\rho, s) < \tau < f(\rho, s)\},$$

null infinity is given in our gauge by

$$\mathcal{J}^\pm = \{\rho > 0, \quad s \in SU(2), \quad \tau = \pm f(\rho, s)\},$$

the *cylinder at space-like infinity* **is given by**

$$I = \{\rho = 0, \quad s \in SU(2), \quad |\tau| < 1\}.$$

It touches $\mathcal{J}^\pm$ at

$$I^\pm = \{\rho = 0, \quad s \in SU(2), \quad \tau = \pm 1\}.$$

The evolution near the cylinder at space-like infinity.

In the following we assume that $\chi_{ab} = 0$ near i.

With $w = (e^\mu{}_k, \hat{\Gamma}_i{}^j{}_k, \hat{L}_{jk})$ and $z = (d^i{}_{jkl})$ the reduced equations read

$$(*) \quad A^\mu(w)\,\partial_\mu\, z = H(w)\, z, \qquad \partial_\tau\, w = F(w, z),$$

$$A^0 \text{ positive definite on } \hat{B}, \quad {}^t\bar{A}^\mu = A^\mu.$$

They can be extended smoothly, as symmetric hyperbolic system, across I into the domain where $\rho < 0$.

Standard results on solution to symmetric hyperbolic systems then imply the existence of a smooth local solution in a neighbourhood of $\hat{B}$ in $\bar{M} = \{\rho \geq 0, \quad s \in SU(2), \quad -f(\rho, s) < \tau < f(\rho, s)\}$, which covers in particular a neighbbourhood of I^0 in I.

The cylinder I is thus generated by conformal geometry (as a limit of conformal geodesics) and the field equations.

The details of the reduced equations give

$$A^\rho = 0 \quad \text{on} \quad I.$$

This implies that the functions

$$u^p = \partial^p_\rho\, u|_I, \;\; p = 0, 1, 2, \ldots,$$

satisfy *interior transport equations* on I and are determined by data on I^0.

Expanding $u^p = u^p(\tau, s)$ in a suitable function system on $SU(2)$ gives for the expansion coefficients $u^{'p}$ a hierarchy of ODE's along the curves $]-1, 1[\ni \tau \rightarrow (\tau, s) \in I$, which in principle can be solved explicitly.

The operators do not depend on the data, the right hand side of the equation for u^{p+1} depends quadratically on $u^0, \ldots, u^p$.

The calculation of $A^\mu(u)$ on I gives in particular

$$A^\tau = diag(1 + \tau, 2, 2, 2, 1 - \tau) \quad \text{on} \quad I.$$

The equations thus degenerate on $I^\pm$ where $\tau = \pm 1$.

This degeneracy pinpoints the central difficulty of the subject.

A conjecture.

The expansion coefficients u'^p which have been calculated so far have in general terms of the form

$$\left(\frac{1-\tau}{2}\right)^{p-k+2}\left(\frac{1+\tau}{2}\right)^{p+k-2}\left\{e+f\int_0^\tau\frac{d\sigma}{(1-\sigma)^{p-k+3}(1+\sigma)^{p+k-1}}\right\}$$

with constants of integration e, f which follow from the data at I^0. The u^p thus develop logarithmic singularities at I^+. The non-linearity of the equations will also imply terms of the form $(1-\tau)^k\log^j(1-\tau)$.

Inspection of the initial data for u'^p on I^0 shows:

The logarithmic terms observed so far in general do not occur in u'^p for $p \leq q_* + 2$ if and only if h satisfies the *regularity condition*

$$\mathcal{S}\left(D_{i_1},\ldots,D_{i_q}\,B_{jk}(i)\right)=0\quad\textbf{for}\quad q=0,1,\ldots,q_*,$$

where D denotes the h-connection, $B_{jk}=1/2\,\epsilon_j{}^{il}\,D_i\,L_{lk}$ the h-Cotton tensor, and $\mathcal{S}$ means: 'take the symmetric trace-free part of ...'.

The conditions are conformally invariant.

Static solutions satisfy these conditions for all q_*.

There exists a large class of data on S^3 which satisfy these conditions.

Conjecture: *There exists an integer $k_* \geq 0$ such that for given $k \geq k_*$ the time evolution of an asymptotically euclidean, time-symmetric, conformally smooth initial data set admits a conformal extension to null infinity of class C^k near space-like infinity, if the regularity condition holds for a certain integer $q_* = q_*(k)$.*

If this is true, the results on the hyperbolidal initial value problem will imply the existence of a large class of asymptotically simple solutions with a smooth conformal structure at null infinity.

The subconjecture.

The calculations seem to hint at an unexpected hidden property of Einstein's equations.

Subconjecture: *If the regularity condition holds for a given* $q_* \geq 0$*, the functions* u^p*,* $p \leq q_* + 2$*, extend smoothly to* $I^{\pm}$.

J. Kánnár, H.F., JMP 2000: verification for $p \leq 3$**.**
J. Valiente Kroon, 2002: verification for $p \leq 4$ **for data which are conformally flat near** i**.**

The analysis will provide an insight into the field equations which goes beyond their conformal properties and their hyperbolicity. It will involve their structure at all orders.

The calculations underlying the analysis provide interesting information on the field and physically relevant quantities near space-like infinity. Examples (assuming the correctnes of the conjecture):

J. Kánnár, H.F., JMP 2000: relate the present gauge to the Bondi gauge and derive a formula for the Newman-Penrose constants

$$G_k = \int_{cut(\mathcal{J}^+)} {}_2\bar{Y}_{2,k}\, \Psi^1_0 \sin\vartheta\, d\vartheta\, d\phi, \quad k = -2, \ldots, 2,$$

in terms of the initial data on S **near** i **which is of the general form**

mass $\times$ *quadrupole moment - (dipole moment)*^{2}

and generalizes a formula derived by Newman and Penrose for static solutions. This determines $\phi_{abcd}(i^+)$ **in terms of initial data at** I^0 **if the solution admits a regular point** i^+**.**

J. Valiente Kroon, arXiv:gr-qc/0206050, June 2002: the calculation of u^p **for** $p \leq 5$ **gives a quite unexpected direct relation between the Bondi mass and the Newman-Penrose coefficients. For solutions arising from data which are conformally flat near** i **the Bondi mass has in terms of a Bondi retarded time** u **an expansion of the form**

$$m_B = m_{ADM} + (2^{7/2} \sum_{k=-2}^{2} |G_k|)\, u^{-7} + O(u^{-8}) \ \text{ as } \ u \to -\infty.$$

Control on behaviour near $\mathcal{J}^+$.

Together these estimates give

$$\int_{N_t}\Big(\sum_{q'+p'+|\alpha|\leq m}|D^{q',p',\alpha}(\partial^p_\rho\phi_k)|^2\Big)\,d\tau\,d\rho\,d\mu \leq$$

$$C\sum_{k=0}^{4}\int_{S_0}\Big(\sum_{q'+p'+|\alpha|\leq m}|D^{q',p',\alpha}(\partial^p_\rho\phi_k)|^2\Big)\,d\rho\,d\mu,$$

$$k = 0,\ldots,4, \quad 0 \leq t < 1,$$

where C depends on p and m but not on $t \in [0,1[$.

Sobolev embedding results imply

$$\partial^p_\rho\,\phi_k \in C^{j,\lambda}(N_1) \quad \textbf{for} \quad p \geq j+6, \quad 0 < \lambda \leq 1/2.$$

Writing $J: f \to J(f) = \int_0^\rho f(\tau, r, s)\,d\,r$ we get by integration

$$\phi_k = \sum_{p=0}^{p-1}\frac{1}{p'\,!}\,\phi_k^{p'}\,\rho^{p'} + J^p(\partial^p_\rho\phi_k) \quad \textbf{on} \quad N_1 \quad \textbf{for} \quad p \geq j+6,$$

resp.

$$\phi_k - \sum_{p'=0}^{p-1}\frac{1}{p'\,!}\,\phi_k^{p'}\,\rho^{p'} \in C^{j,\lambda}(N_1) \quad \textbf{for} \quad p \geq j+6,$$

where the $\phi_k^{p'}(\tau, s) = \partial^{p'}_\rho\,\phi_k|_I$, which are obtained by integrating the transport equations on I, are extended to N_1 as ρ-independent functions.

Thus the overdeterminedness of the equations and the special form of the differential operators, including the existence of the transport equations on I, allow us to get complete control on ϕ_k near $\mathcal{J}^+$.

If the linearized regularity condition is satisfied for all q_*, the $\phi_k^{p'}$ extend smoothly to $I^\pm = \{\rho = 0, \tau = \pm 1, s \in SU(2)\}$, and $\phi_k \in C^{j,\lambda}(N_1)$ for all j.
Since the ϕ_k^p do in general develop logarithmic singularities, the same can be expected of ϕ_k in the non-linear case.

Can the argument be extended to the non-linear case ?