Black Holes, the Penrose Conjecture, and Quasi-Local Mass

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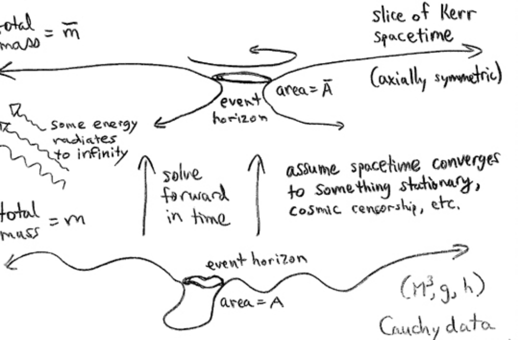
What is the Penrose Conjecture?

Physical motivation (R. Penrose, 1973)



y the Hawking Area Theorem, $\bar{A} \geq A$.

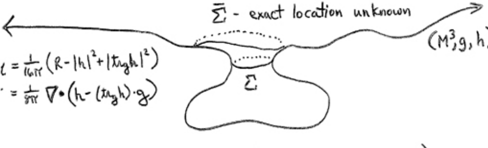
llowing for some energy to radiate to infinity, we hould have $m \geq \bar{m}$. Then since $\bar{m} \geq \sqrt{\bar{A}/16\pi}$ n Kerr spacetimes, we should have

$$m \geq \sqrt{A/16\pi}$$

or the original arbitrary data (M^3, g, h).

here's one problem though: given (M^3, g, h) here's no general procedure for determining the cation of the event horizon (without actually volving the data maximally in the future).

olution: An apparent horizon is defined by the cal geometry and implies the existence of an event orizon outside of it.



<u>he Penrose Conjecture</u> (Wald formulation)

et (M^3, g, h) be complete, asymptotically flat, pace-like Cauchy data with $\mu \geq |J|$ and an pparent horizon Σ satisfying $H_\Sigma = tr_\Sigma h$. Then

$$m \geq \sqrt{\frac{A}{16\pi}}$$

here m is the total ADM mass and A is the inimum area required for a surface to enclose Σ.

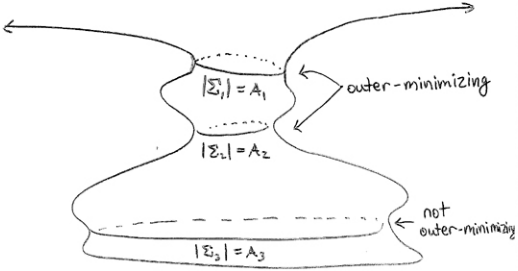
he Riemannian Penrose Inequality

et (M^3, g) be a complete, asymptotically flat iemannian manifold with total mass m and calar curvature $R \geq 0$. Then

$$m \geq \sqrt{A/16\pi}$$

here A is the area of any outer-minimizing inimal surface Σ^2 in M^3.

ote that, by definition, the minimum area needed to nclose an outer-minimizing surface of area A is A.



97 – Huisken-Ilmanen (when Σ is connected)

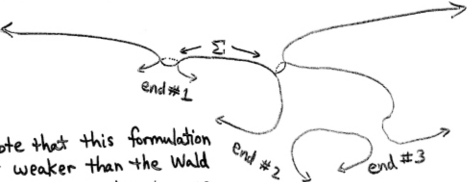
99 – B.

he Penrose Conjecture (B. formulation)

et (M^3, g, h) be complete, asymptotically flat (with multiple ends) with $\mu \geq |J|$ and total mass m in a chosen end. Then

$$m \geq \sqrt{\frac{A}{16\pi}}$$

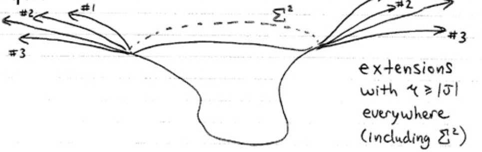
where A is the area of a globally-minimizing minimal surface Σ^2 in M^3 containing all of the ends (except the chosen one).

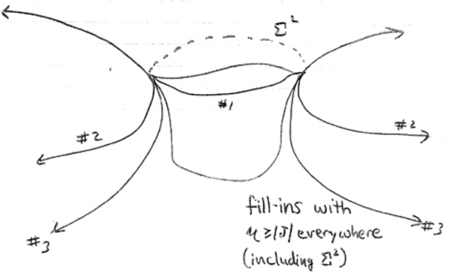


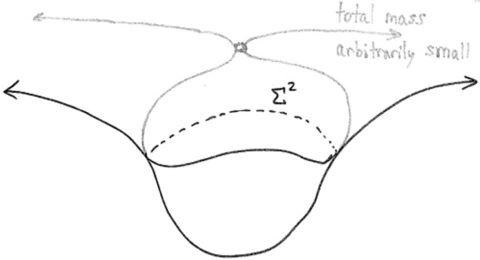
ote that this formulation weaker than the Wald rmulation. Also, we are ssuming that each end has a large trapped sphere S with $H_S < \mathrm{tr}_S h$ (which follows from sufficient decay of h) $\theta(1/r^2)$. he advantage of this formulation is that there is no pparent horizon. Hence, the time-symmetric and general ersions of this conjecture differ only by replacing $R \geq 0$ with $\mu \geq |J|$.

Quasi-Local Mass

Let's assume that the Penrose Conjecture is true. Then we can define some quasi-local mass functionals with some great properties. Given $\Sigma^2 \subset (M^3, g, h)$, consider all other asymptotically flat "extensions" (in which Σ^2 remains outer-minimizing) and geodesically complete "fill-ins" with $\mu \geq |J|$. (M^3, g, h)







Note that the yellow extension is not legal since Σ^2 is no longer outer-minimizing. The fact that this extension is not allowed is critical, as we will see in a moment.

Define the Bartnik outer mass of Σ^2 to be

$$m_{outer}(\Sigma) = \inf_{\mathcal{E}} \left\{ m(\mathcal{E}) \;\middle|\; \begin{array}{l} m \text{ is the total mass of the af.} \\ \text{extension } \mathcal{E} \text{ with } \mu \geq |J| \text{ in which} \\ \Sigma \text{ remains outer-minimizing.} \end{array} \right\}$$

and the inner mass of Σ^2 to be (Note monotonicity of both functionals!)

$$m_{inner}(\Sigma) = \sup_{\mathcal{F}} \left\{ \sqrt{\frac{A(\mathcal{F})}{16\pi}} \;\middle|\; \begin{array}{l} A \text{ is the minimum area required} \\ \text{to contain all of the ends of } \mathcal{F}, \\ \text{which has } \mu \geq |J|. \end{array} \right\}$$

Note that any quasi-local mass function which is always less than or equal to the ADM mass must be less than or equal to the Bartnik outer mass. Also, to be "compatible with Penrose," it must be greater than or equal to the inner mass.

Corollary to Penrose Conjecture (either formulation)

For all outer-minimizing $\Sigma^2 \subset (M^3, g, h)$,

$$m_{outer}(\Sigma^2) \geq m_{inner}(\Sigma^2).$$

Proof: Consider any extension and fill-in at the same time. Since Σ is outer-minimizing, a global area minimizer always lies inside Σ. Then take the sup and inf on Penrose.

Theorem Suppose Σ_2 encloses Σ_1 and both are outer-minimizing in (M^3, g, h). Then

$$m_{inner}(\Sigma_2) \geq m_{inner}(\Sigma_1)$$
$$m_{outer}(\Sigma_2) \geq m_{outer}(\Sigma_1).$$

Proof. Any fill-in of Σ_1 is also a fill-in of Σ_2, so the first inequality is obvious. For the second, though, we need to show that Σ_1 is still outer-minimizing in any extension of Σ_2 in which Σ_2 is outer-minimizing.

$$|\Sigma| = |\partial D| \geq |\partial(D \cap D_2)| \geq |\partial D_1| = |\Sigma_1|.$$

Thus, any surface ~~on~~ Σ enclosing Σ_1 has greater or equal area.

Possible Point of View: In the same way that gravitational energy is not given by local expressions, perhaps it is too much to ask to assign a single number to the mass of a given region. That is, perhaps gravitational energy refuses to say whether or not it is in a given region. Instead, let's define the quasi-local mass of Σ^2 in (M^3, g, h) to be the <u>interval</u> $[m_{inner}, m_{outer}](\Sigma)$.

<u>Definition</u>: Let the total outer mass, total inner mass be

$$m_{outer} = \sup_{\Sigma} \{ m_{outer}(\Sigma) \mid \Sigma \text{ outerminimizing} \}$$

$$m_{inner} = \sup_{\Sigma} \{ m_{inner}(\Sigma) \mid \Sigma \text{ outerminimizing} \}$$

which is the same thing as taking a limit as Σ goes to infinity since $m_{outer}(\cdot)$ and $m_{inner}(\cdot)$ are non-decreasing with respect to enclosure.

Conjecture: In an asymptotically flat (M^3, g, h),

$$m_{ADM} = m_{outer} = m_{inner}.$$

But what local geometric information on $\Sigma^2 \subset (M^3, g, h)$ is $m_{inner}(\Sigma)$ and $m_{outer}(\Sigma)$ really based?

Answer: $g|_\Sigma$, the in-going and out-going null convergences θ_-, θ_+, and the connection on the normal bundle of Σ. This data on Σ is called the Bartnik data.

Note that $\theta_\pm = H^M_\Sigma \pm tr_\Sigma h$ in terms of the slice (M^3, g, h), or equivalently $\theta_\pm = \langle \vec{H}^N_\Sigma, n_\pm \rangle$ where $\vec{H}^N_\Sigma$ is the mean curvature vector of Σ in the space-time N and n_-, n_+ are the in-going and out-going null vectors (future pointing) $\perp \Sigma$.

Equivalently, then, the Bartnik data is $g|_\Sigma$, $\vec{H}^N_\Sigma \subset$ the normal bundle to Σ, and the connection on the normal bundle of Σ. Note that this data is independent of any particular slice (M^3, g, h).

Definition We say that Bartnik data on Σ is "legal" if there exists (M^3, g, h) which is geodesically complete and asymptotically flat with $\mu \geq |J|$ such that the Bartnik data is induced from an inclusion of $\Sigma \subset M^3$ in which Σ is outer-minimizing.

Point: $m_{outer}(\Sigma)$ and $m_{inner}(\Sigma)$ are well-defined for all Σ with legal Bartnik data. Thus, if we allow (M^3, g, h) to have <u>any</u> asymptotics, the total inner and outer masses $[m_{inner}, m_{outer}]$ are defined in $\mathbb{R} \cup \{\infty\}$ as long as there is at least one $\Sigma \subset M^3$ with legal Bartnik data.

Mark Aarons, last week, asked, "When does $m_{inner} \neq m_{outer}$?" Good question.

Conjecture (1st attempt)

Suppose (M^3, g, h) has $\mu \geq |J|$, $g > 0$, and a sequence $\{\Sigma_i\}_{i=1}^{\infty}$, $\Sigma_i = \partial D_i \subset M$ with legal Bartnik data, $\bigcup_{i=1}^{\infty} D_i = M$, and $\lim_{i \to \infty} |\Sigma_i| = \infty$. Then

$$m_{inner} = m_{outer} \in \mathbb{R} \cup \{\infty\}.$$

Point: Crazy asymptotics should lead to both m_{inner} and m_{outer} going to ∞. Either of them being finite should be quite restrictive and imply something like an approximate embedding of (M^3, g, h) outside Σ_i into Schwarzschild. In this case we expect m_{inner} and m_{outer} to be equal.

Definition

When $m_{inner} = m_{outer}$, we define m_{total} to be this value. Otherwise, we say that m_{total} is undefined.

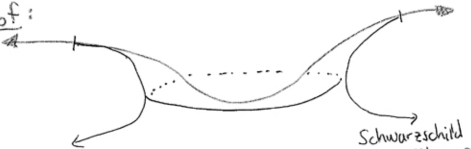
All of these definitions also have a Riemannian version (meaning $h \equiv 0$). Then (M^3, g) is required to have $R \geq 0$, and we only take infs and sups over data with $h \equiv 0$. Define the Riemannian inner and outer quasi-local mass functions to be

$$m^{\circ}_{inner}(\Sigma), \quad m^{\circ}_{outer}(\Sigma).$$

Theorem Suppose (M^3, g) has $R \geq 0$ and is harmonically flat at ∞. Then

$$m_{ADM} = m^{\circ}_{inner} = m^{\circ}_{outer} = m^{\circ}_{total}.$$

Proof:



Schwarzschild slice with mass $(m_{ADM} - \varepsilon)$.

By Riemannian Penrose Inequality,

$$m_{ADM} \geq m^{\circ}_{outer} \geq m^{\circ}_{inner} \geq m_{ADM} - \varepsilon. \qquad \text{Send } \varepsilon \to 0. \qquad \blacksquare$$