Multi Black Hole Configurations

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Introduction

A **conjecture** for the asymptotic behavior of solutions of the Einstein Vacuum Equations:

> *For any asymptotically flat initial data the solution settles to one consisting of finitely many Kerr Black Holes moving away from each other with constant velocity.*

Intuitive idea: Any feature other than mass and angular momentum is either radiated away, or swallowed by the black holes.

First task: Find all asymptotically flat stationary solutions of the Einstein Vacuum Equations.

Definition: We say that a strongly asymptotically flat spacetime (M, g) is *stationary* if it admits an isometric $\mathbb{R}$-action whose generator K is timelike near spacelike infinity.

If we omit the last clause in the definition above, then we rule out the Kerr solution.

Stationary Solutions

Let (M, g) be a stationary solution of the Einstein Vacuum Equations, and:

$$X = -g(K, K), \qquad \omega = *(K^\flat \wedge dK^\flat).$$

The Einstein Vacuum Equations imply $d\omega = 0$, hence ($\sim$ topological considerations), $\exists Y$ such that $dY = \omega$.

Now, let $\varphi_\pm = (\pm X, Y)$, $M_\pm = \{\pm g(X, X) < 0\}$, then

$$\varphi_\pm \colon M_\pm \to (\mathbb{H}^2, h)$$

is a *wave map*, where $\mathbb{H}^2 = \{(X, Y), X > 0\}$, and h is the hyperbolic metric:

$$ds_h^2 = \frac{dX^2 + dY^2}{X^2}.$$

Let $Q_\pm$ be the quotient of $M_\pm$ by the given $\mathbb{R}$-action, and let $g_\pm = Xg \pm K \otimes K$ be the *quotient metric* on $Q_\pm$ induced by K. Then

- $\varphi_\pm \colon (Q_\pm, g_\pm) \to (\mathbb{H}^2, h)$ is a $\begin{pmatrix}\text{harmonic}\\ \text{wave}\end{pmatrix}$-map;
- $2G(g_\pm) = T(\varphi_\pm)$.

Stationary Axisymmetric Solutions

Definition: We say that a strongly asymptotically flat spacetime (M, g) is *stationary axisymmetric* if it admits an isometric $\mathbb{R}\times U(1)$-action whose orbits are everywhere timelike.

Let (M, g) be a stationary axisymmetric solution and take:

- K to be the generator of the $U(1)$-action
- T to be the only generator which is timelike at infinity;
- $X = g(K, K)$, V=-g(T,T); W=g(K,T);
- $\rho^2 = XV + W^2$ minus the area element of the orbits.

The metric on the orbits is $-V\,dt^2 + 2W\,dt\,d\phi + X\,d\phi^2$. The twists are:

$$\sigma = *(K^\flat \wedge T^\flat \wedge dK^\flat), \qquad \tau = *(K^\flat \wedge T^\flat \wedge dT^\flat).$$

The Einstein Vacuum Equations again imply $d\sigma = d\tau = 0$, hence these scalars are constants. Thus, if the *axis*

$$\mathcal{A} = \{K = 0\}$$

is not empty, then $\sigma = \tau = 0$, and the quotient space Q of M can be represented by an integral surface of the distribution $\perp$ to the orbits.

Let γ be the metric of Q. Then

- $\Delta_\gamma \rho = 0$ and $\nabla \rho \neq 0$ outside $\mathcal{A}$, thus ρ is an *isothermal* coordinate on Q.
- $ds_\gamma^2 = e^{2\lambda}\left(d\rho^2 + dz^2\right)$ where z is a *conjugate coordinate*.

Let δ be the flat metric $\delta = d\rho^2 + dz^2 + \rho^2 d\phi^2$ on $Q \times S^1$, then:

$$\varphi = (X, Y) \colon (Q \times S^1, \delta) \to (\mathbb{H}^2, h), \qquad \varphi_\phi = 0.$$

is a harmonic map. Once X and Y are known, then λ can be found by quadrature:

$$(\lambda_\rho, \lambda_z) = \text{given in terms of } X, Y.$$

Boundary Conditions.

- Regularity $\mathcal{A}$ implies: X/ρ^2 is bounded on $\mathcal{A}$ away from the end points.
- Asymptotic flatness implies: $X/\rho^2 \to 1$ at infinity.
- *Apparent horizons*, corresponds to loci where $\rho = 0$ and $X \neq 0$.
- Y is constant on each component of $\mathcal{A}$.

The Boundary Value Problem

Provisional

Fix N (the number of black holes), let $\mathcal{A}$ be the z-axis with N disjoint finite intervals removed, and pick $N+1$ constants y_i, $i = 0, \ldots, N$. Find a harmonic map $\varphi = (X, Y)\colon (\mathbb{R}^3 \setminus \mathcal{A}, \delta) \to (\mathbb{H}^2, h)$:

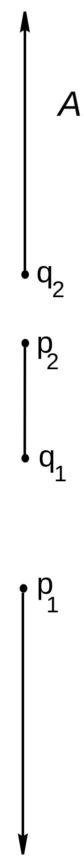
$$\Delta \log X = -\frac{|\nabla Y|^2}{X^2}, \qquad \operatorname{div}\left(\frac{\nabla Y}{X^2}\right) = 0.$$

which satisfies the (singular) boundary conditions:

- X is *asymptotic* to ρ^2 near $\mathcal{A}$ away from the end points.
- $X/\rho^2 \to 1$ at infinity.
- $Y = y_i$ on $\mathcal{A}_i$ (the connected components of $\mathcal{A}$).

Note: the number of free parameters = the number of physical parameters:

$$\underbrace{2N}_{\text{end points}} + \underbrace{N+1}_{\text{constants } y_i} - \underbrace{2}_{\text{symmetries}} = \underbrace{N}_{\text{masses}} + \underbrace{N}_{\text{angular momenta}} + \underbrace{N-1}_{\text{distances.}}$$



A
q2
p2
q1
p1

The Linear Problem

If we set $Y = 0$ (ruling out rotation) we get a linear problem for:

$$u = \log X.$$

Indeed, the map $\varphi = (e^u, 0)\colon \mathbb{R}^3 \setminus \mathcal{A} \to (\mathbb{H}^2, h)$ is harmonic if and only if the function u is harmonic. Note that φ maps into a geodesic of $(\mathbb{H}^2, h)$.

For the Schwarzschild solution, u is the normalized Newtonian potential of a uniform unit charge on the axis $\mathcal{A}$:

$$u = \log(r_p - z_p + z) + \log(r_q + z_q - z),$$

where p and q are the end point of the axis. It is now straightforward to find the solution of the boundary problem for arbitrary N. This was carried out by H. Weyl (1917).

We write this solution $u = \sum_{i=1}^{N+1} u_i$, where u_i is the potential of a uniform unit charge on $\mathcal{A}_i$.

Question: Does this yield new static solutions.
Answer: No. (More on this later).

Harmonic Maps with Prescribed Singularities

- (M, g) is a non-compact complete Riemannian manifold.
- $\mathcal{A} \subset M$ has co-d ≥ 2, and connected components $\mathcal{A}_i$.
- (N, h) is a Hadamard manifold with $-b^2 < \kappa < -a^2$.

Definition: We say that two maps $\varphi, \psi \colon (M \setminus \mathcal{A}, g) \to (N, h)$ are *asymptotic* near a component $\mathcal{A}_0 \subset \mathcal{A}$ if $\mathrm{dist}_N(\varphi, \psi)$ is bounded in a neighborhood of $\mathcal{A}_0$. We say that φ and ψ are asymptotic near infinity if $\mathrm{dist}_N(\varphi, \psi) \to 0$ at infinity.

Note: If γ is a geodesic of (N, h), and $u \colon M \setminus \mathcal{A} \to \mathbb{R}$ is a harmonic function with $u \to \infty$ on $\mathcal{A}$, then $\varphi \circ \gamma \colon (M \setminus \mathcal{A}, g) \to (N, h)$ is a harmonic map with $\varphi \to \gamma(+\infty)$ on $\mathcal{A}$.

Theorem. *Let $\varphi_i = \gamma_i \circ u_i \colon (M \setminus \mathcal{A}_i, g) \to (N, h)$ be a harmonic maps as above, and suppose that there is $\psi \colon (M \setminus \mathcal{A}, g) \to (N, h)$ such that:*

(i) *ψ is asymptotic to φ_i near $\mathcal{A}_i$ for each i;*

(ii) *$\exists v$ bounded on M with $v \to 0$ at infinity and $|\tau(\psi)| \leq \Delta_g v$.*

Then there exists a unique harmonic map $\varphi \colon (M \setminus \mathcal{A}, g) \to (N, h)$ which is asymptotic to ψ near $\mathcal{A}$ and at infinity.

The Boundary Value Problem
Revisited

Recall that $\varphi_i = (e^{u_i}, y_i)\colon (\mathbb{R}^3 \setminus \mathcal{A}_i, \delta) \to (\mathbb{H}^2, h)$ are harmonic maps as above. Thus, it remains to construct ψ satisfying (i) and (ii).

Note: In $\mathbb{R}^3$, we can take

$$v = -(1 + r^2)^{-\varepsilon}, \qquad 0 < \varepsilon < 1/2,$$

to get

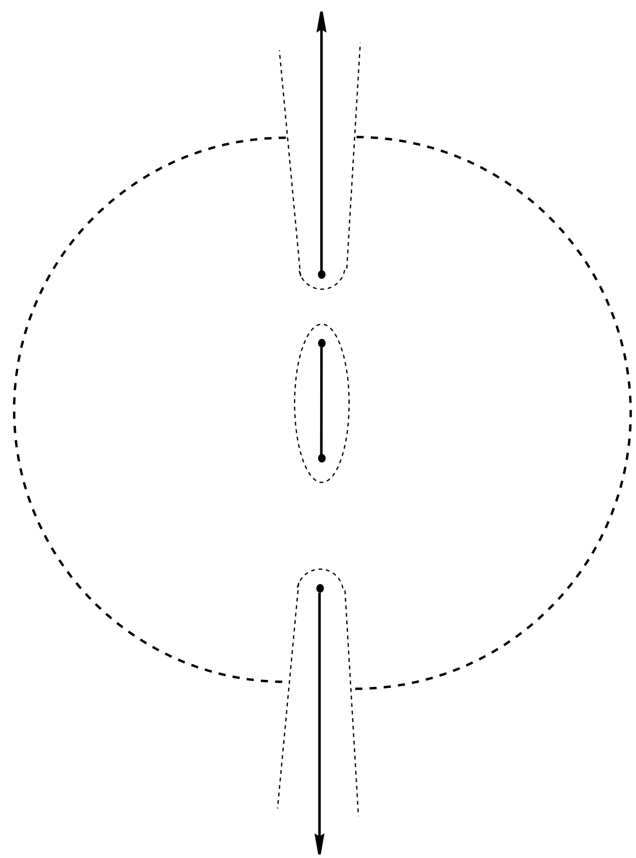
$$\Delta v \geq c(1 + r^2)^{-1-\varepsilon}.$$

Thus, in order to satisfy (ii) it suffices to find verify

$$|\tau(\psi)| = O(r^{-2-\varepsilon}).$$

Lemma. *Suppose $\Delta u = 0$, with $u \to \infty$ on $\mathcal{A}$ and $u - 2\log\rho \to 0$ at infinity, and suppose that Y is constant near each component of $\mathcal{A}$ and depends only on the polar angle outside a sufficiently large ball. Then, if $\psi = (e^u, Y)$, we have $|\tau(\psi)| = O(r^{-3})$.*

Conclusion: Let (M, g) be a solution of the Einstein Vacuum Equations which is globally hyperbolic, asymptotically flat, stationary axisymmetric, and such that every component of the event horizon is non-degenerate. Then the corresponding reduced boundary value problem has a unique solution.



Generalizations

If we replace the Einstein Vacuum Equations with the Einstein Maxwell Equations, then we obtain the same problem with the *complex hyperbolic plane* $\mathbb{H}^2_{\mathbb{C}}$ as target instead of the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$. The metric of $\mathbb{H}^2_{\mathbb{C}} = \{(X, Y, E, B) \colon\ X > 0\}$ can be written as:

$$ds^2 = \frac{dX^2 + (dY + B\,dE - E\,dB)^2}{X^2} + \frac{dE^2 + dB^2}{X}.$$

If we consider instead the Einstein Abelian-Yang-Mills Equations (with structure group $\mathbb{R}^k$:

$$\text{Ric} - \frac{1}{2} R\, g = 2T,$$
$$F = dA, \qquad d * F = 0,$$
$$T(X, Y) = \text{tr}(i_X F \cdot i_Y F + i_X * F \cdot i_Y * F).$$

then we obtain the same problem with the higher dimensional *complex hyperbolic space* $\mathbb{H}^{k+1}_{\mathbb{C}}$ as target.

Question: Is there a matter model for which the target space is the *quaternionic hyperbolic space* $\mathbb{H}^k_{\mathbb{H}}$?

The Converse

Question: Given a solution of the reduced boundary value problem, can we reconstruct a solution of the Einstein Vacuum Equations (or Einstein Matter Equations).

Let

$$dW = \rho \frac{*dY}{X^2}, \qquad d\lambda = \text{given in terms of } X \text{ and } Y, \qquad V = \frac{\rho^2 - W^2}{X}.$$

Then the spacetime $(\mathbb{R} \times \mathbb{R}^3, g)$ with the metric:

$$ds^2 = -V\,dt^2 + 2W\,dt\,d\phi + X\,d\phi^2 + e^{2\lambda}(d\rho^2 + dz^2),$$

is a globally hyperbolic, asymptotically flat, stationary axisymmetric solution of the Einstein Vacuum Equations with non-degenerate horizon, *provided*:

$$\frac{X^{1/2}}{\rho e^{\lambda}} \to 1, \qquad \text{on } \mathcal{A}.$$

This condition expresses the absence of *conical singularities* on $\mathcal{A}$ in the metric g. Indeed,

$$\frac{X^{1/2}}{\int_0^\rho e^{\lambda}\,d\rho} \sim \frac{X^{1/2}}{\rho e^{\lambda}}, \qquad \rho \to 0.$$

Known Cases

1. **The Linear Case** (H. Weyl, 1917). In this case, the angle deficiency can be computed. Weyl also related this angle deficiency to the *gravitational force*. In the limit where the distance tends to infinity, the force is asymptotic to the Newtonian gravitational force.

2. **Symmetric Case** (Y. Li and G. Tian, 1991). In this case, the authors assume the existence of an involutive symmetry $z \mapsto -z$. Thus, for example if $N = 2$, this will be the case if the masses are equal, and if the angular momenta are of equal magnitude and opposite sign. They then show that the force is attractive.

3. **Close and Near Extreme Case** (G. W., 1994). It is shown (in terms of the free parameters) that provided the distance between two adjacent horizons is small enough, if the size of the interval corresponding to one of the horizons tends to zero (black hole becomes *extreme*), then the force tends to infinity.

4. **Symmetric Case** (Neugebauer-Kreuser, 1999).