The Penrose proposal, problems and results

Gravitational radiation ?

Einstein (1915) and textbooks discuss "gravitational radiation" in terms of the linearized Einstein equations.

Pirani, Trautman, Sachs, Bondi, Newman, Penrose , ... (ca. 1957 - 1962) : Does there exists a concept of radiation based on the non-linear theory ?

Maxwell theory: Perturbations travel inside or on the null cone. Need to go far out for perturbations to develop into 'waves'. Suggests:

Analyse the propagation of gravitational fields on outgoing null hypersurfaces. Follow the propagation far out.

Where is 'far out', what does it look like, how do we identify 'radiation' if there is no background space ? How do we analyse these questions ?

Specialize: Consider systems of stars, generating the perturbation, which are "far away" from other star systems ("isolated system").

Idealize: Assume that the field approaches in some sense the Minkowski field as the affine parameter $r \to \infty$ along the outgoing null geodesics.

Hope: Higher order quantities have a limit which can be interpreted as representing 'gravitational radiation'.

Gravitational Radiation !

GUESS: $\Psi_0 = O(r^{-5})$ **in a uniform way as** $r \to \infty$**.**

Formal expansion (in $\frac{1}{r}$**) type analysis gives:**

$\Psi_k = O(r^{k-5})$ **as** $r \to \infty$ **("Sachs peeling").**
$\lim_{r\to\infty} r\,\Psi_4$ **represents the "radiation field".**
The 'Bondi-mass' satisfies a 'mass-loss formula' (and is positive). 'Gravitational waves carry only positive energy'.

In these considerations the null cone structure and associated structures (null hypersurfaces, null geodesics etc.) played a critical role. As an object in itself the null cone structure is arkward to handle. However, in terms of the equivalent 'conformal structure', it can be conveniently analysed.

The critical role of the null cone structure in the studies above was highlighted by the following observation:

Penrose (1963): "The asymptotic behaviour postulated resp. derived above can be characterized geometrically solely in terms of the asymptotic behaviour of the conformal structure".

The more precise statement/proposal following this observation raised some controversy.

The following discussion will largely be concerned with the nature of the proposal, its problems, and the attempt to provide a firm foundation for it.

But it will also show that the consequent abstract analysis of the proposal has practical consequences.

Conformal extensions I.

Construct 'conformal extensions' for the simply connected, conformally flat standard solutions to $Ric[\tilde{g}] = \lambda\, \tilde{g}$ by suitable embeddings.

Target in all 3 cases: the 'Einstein cosmos' $(\bar{M}, g)$ with $\bar{M} = I\!R \times S^3$, $g = d\,s^2 - d\,\omega^2 = d\,s^2 - (d\,\chi^2 + \sin^2\chi\, d\,\sigma^2)$.
(Standard line elements: $d\,\omega^2$ on S^3, $d\,\sigma^2$ on S^2)

$\lambda < 0$, de Sitter space: $\tilde{M} = I\!R \times S^3$, $\tilde{g} = dt^2 - \cosh^2 t\, d\,\omega^2$.

The map $\tilde{M} \ni (t,\ \vartheta) \stackrel{\Phi}{\rightarrow} (s = \arctan e^t - \frac{\pi}{4},\ \vartheta) \in \tilde{M}' \subset \bar{M}$ defines a diffeomorphism onto $\Phi(\tilde{M}) = \tilde{M}' =]-\pi/4, \pi/4[\times S^3$, in fact a conformal embedding: holds $\Omega^2\, \Phi^{-1*}\tilde{g} = g$, $\Omega = \cos(2\,s) > 0$ on $\tilde{M}'$.

Observations:

$\Phi(\tilde{M})$ has a C^∞ boundary $\mathcal{J} \equiv \partial\,(\Phi(\tilde{M})) = \mathcal{J}^+ \cup \mathcal{J}^-$ in M,

Ω and $g = \Omega^2\, \Phi^{-1*}\tilde{g}$ extend smoothly to $M = \tilde{M}' \cup \mathcal{J}$,

$\Omega = 0$, $d\,\Omega \neq 0$ on $\mathcal{J}^\pm = \{s = \pm\pi/4\} \sim S^3$.

The 'physical' space-time is finite, the 'conformal boundary at null (and time-like) infinity' at a finite location with respect to the 'conformal metric' g.

Convenient analysis of asymptotic behaviour of Maxwell fields.

Convenient analysis of global causal space-time structure, observe e.g. that $\mathcal{J}$ is space-like.

Conformal extensions II.

$\lambda > 0$, **anti-de Sitter covering space:**
$\tilde{M} = I\!R^4$, $\tilde{g} = \cosh^2 r\, dt^2 - (dr^2 + \sinh^2 r\, d\sigma^2))$.

The map $\tilde{M} \ni (t, r, \vartheta) \xrightarrow{\Phi} (s = t, \chi = 2\ \mathrm{arctan}\,(e^r) - \frac{\pi}{2}, \vartheta) \in \tilde{M}' \subset \bar{M}$,
defines a diffeomorphism onto $\Phi(\tilde{M}) = \tilde{M}' = \{\chi < \pi/2\}$, **in fact a conformal embedding: holds** $\Omega^2\, \Phi^{-1*}\tilde{g} = g$, $\Omega = \cos\chi > 0$ **on** $\tilde{M}'$

Observations:

$\Phi(\tilde{M})$ **has a** C^∞ **boundary** $\mathcal{J}$ **in** M,

Ω **and** $g = \Omega^2\, \Phi^{-1*}\tilde{g}$ **extend smoothly to** $M = \tilde{M}' \cup \mathcal{J}$,

$\Omega = 0$, $d\,\Omega \neq 0$ **on** $\mathcal{J} = \{\chi = \pi/2\} \sim I\!R \times S^2$.

The conformal space-time is finite in space-like directions, the conformal boundary at null (and space-like) infinity is at spatially finite location with respect to the conformal metric g.

The conformal boundary $\mathcal{J}$ **is time-like.**
$\exists$ **non-vanishing solutions of wave equations in AdS which vanish in a neighbourhood of the slice** $\{t = 0\}$.
$\nexists$ **Cauchy hypersurface in AdS ('not globally hyperbolic').**

Conformal extensions III.

$\lambda = 0$**: Minkowski space** $(\tilde{M} = I\!R^4, \tilde{g} = dt^2 - (dr^2 + r^2\, d\,\sigma^2)$

The map $\Phi : \tilde{M} \to \tilde{M}' = \{|s \pm \chi| < \pi, \;\; \chi \geq 0\} \subset \bar{M}$ **with inverse**

$$\Phi^{-1}: \qquad t = \frac{\sin s}{\cos s + \cos \chi}, \qquad r = \frac{\sin \chi}{\cos s + \cos \chi},$$

defines a diffeomorphism of $\tilde{M}$ **onto** $\tilde{M}'$**, in fact a conformal embedding:**
$\Omega^2\, \Phi^{-1*}\tilde{g} = g$**,** $\Omega = \cos\, s + \cos\, \chi > 0$ **on** $\tilde{M}'$**.**

Observations:

'Conformal boundary': $\mathcal{J} \equiv \partial\Phi(\tilde{M}) = \mathcal{J}^- \cup \mathcal{J}^+ \cup \{i^-, i^0, i^+\}$ **with:**

'future (past) null infinity' $\mathcal{J}^\pm = \{\tau \pm \chi = \pm\pi,\, 0 < \chi < \pi\}$**,**
$\Omega = 0, d\,\Omega \neq 0$ **on** $\mathcal{J}^\pm \sim I\!R \times S^2$**, null hypersurfaces,**
'physical' null geodesics acquire endpoints on $\mathcal{J}^\pm$**,**

'future (past) time-like infinity' $i^\pm = \{\chi = 0, \tau = \pm\pi\}$**,**
endpoint of 'physical' time-like geodesics in the future (past).

'space-like infinity' $i^0 = \{\chi = \pi, \tau = 0\}$**,**
endpoint of 'physical' space-like geodesics.

$\Omega = 0, d\,\Omega = 0, Hess(\Omega) \sim g$ **at** $i^\pm$**,** i^0**.**

For detailed investigations near these points sometimes the inversion $x^\mu \to \frac{x^\mu}{x_\nu\, x^\nu}$ **is useful.**

In all three cases:
the conformal metric g**, the conformal factor** Ω**, and derived fields define solutions to the metric conformal field equations which extend smoothly, as solutions, to the complete Einstein cosmos.**

The Penrose proposal I.

Our observations can be summarized in the generalizing definition

Asymptotic simplicity: *A smooth space-time* $(\tilde{M}, \tilde{g})$ *is called asymptotically simple if there exists a smooth oriented, time-oriented, causal space-time* (M, g) *and on* M *a smooth function* Ω *such that:*

(i) M *is a manifold with boundary* $\mathcal{J}$*,*
(ii) $\Omega > 0$ *on* $M \setminus \mathcal{J}$ *and* $\Omega = 0$*,* $d\,\Omega \neq 0$ *on* $\mathcal{J}$*,*
(iii) there exists an embedding Φ *of* $\tilde{M}$ *onto* $\Phi(\tilde{M}) = M \setminus \mathcal{J}$ *such that*

$$\Omega^2\, \Phi^{-1*}\tilde{g} = g,$$

(iv) each null geodesics of $(\tilde{M}, \tilde{g})$ *acquires two distinct endpoints on* $\mathcal{J}$**.**

Purely differential geometric restriction on global structure.

Implies that all null geodesics are complete.

Thus $\mathcal{J}$ **thus represents** *boundary at null infinity*, **generated by ideal endpoints of null geodesics.**

(ii) redundently implies that $\mathcal{J}$ **is a smooth hypersurface,**
(ii) specifies together with (iii) how precisely $\Phi^{-1*}\tilde{g}$ **is to be rescaled to obtain a smooth, non-degenerate metric.**

The definition was motivated by the following novel idea

Penrose (1963, 1965) : *Far fields of isolated systems behave like asymptotically simple space-times in the sense that they can be smoothly extended to null infinity, as indicated above, after suitable conformal rescalings***.**

(vagueness of formulation the lecturer's . . .)

The Penrose proposal II.

The definition should be applied with understanding, it may require modifications: cf. Schwarzschild-Kruskal solution.

We shall in the following not be worried to much with the completeness condition (iv).

The idea brings out the key geometrical structure in previous investigations. It is independent of any distinguished coordinate systems (like 'Bondi coordinates').

It is of practical interest for analysing space-times:

$$complicated\ \ limits \longleftrightarrow local\ \ differential\ \ geometry$$

It is of interest for analysing physical concepts:

$$approximations\ \ in\ \ (\tilde{M}, \tilde{g}) \longleftrightarrow fields\ \ near\ \ \mathcal{J}$$

(Bondi energy momentum, radiation field, ...)

It is of interest for calculating complete space-times numerically by using the conformal field equations

$$artificial\ \ boundaries \longleftrightarrow finite\ \ conformal\ \ spacetime$$

Field equations + asymptotic simplicity have strong implications:
If $\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu}$ near $\mathcal{J}$ then

$$g^{\mu\nu}\, \nabla_\mu\, \Omega\, \nabla_\nu\, \Omega = -\frac{1}{3}\, \lambda, \qquad \nabla_\mu\, \Omega\, C^\mu{}_{\nu\lambda\rho} = 0 \quad \textbf{on}\ \ \mathcal{J}.$$

The sign of the cosmological constant ($\sim$ term of zeroth order) determines the causal nature of the conformal boundary.

If $\lambda \neq 0$, then $C^\mu{}_{\nu\lambda\rho} = 0$ on $\mathcal{J}$.

The Penrose proposal III.

Central and critical: Asymptotic simplicity is in competition with the implications of the quasi-linear, gauge hyperbolic field equations.

Proposal suggests an extremely sharp characterization of the fall-off behaviour of the gravitational fields of isolated systems

No further strengthening possible without implying essential restrictions.

However: is it going too far already ? What are the criteria ?

The question is not whether C^∞ should be replaced by C^k with some large k but whether there exist extensions of class C^k with k large 'enough' such that e.g. the curvature decays to zero at null infinity.

Penrose (1965) obtains the following remarkable result:

- **$(\tilde{M}, \tilde{g})$ is smooth, solves $\tilde{R}_{\mu\nu} = 0$, and admits a conformal extension (M, g, Ω) with M of class C^4, g and Ω of class C^3,**
- **the Weyl spinor Ψ_{abcd} satisfies $\Omega \, \nabla_{ee'} \, \nabla^a{}_{a'} \, \Psi_{abcd} \to 0$ at $\mathcal{J}^+$,**
- **the set of null generators of $\mathcal{J}^+$ is diffeomorphic to S^2**

implies $\Psi_{abcd} \to 0$ on $\mathcal{J}^+$.

Thus, if M is of class C^5, g and Ω are of class C^4 the curvature decays to zero at null infinity.

Is the class of solutions to Einstein's equations which admit such extensions 'sufficiently' rich ?

Construction of extensions I.

How to decide whether a solution $(\tilde{M}, \tilde{g})$ is asymptotically simple ?

In general:
NOT by using the Einstein cosmos or the inversion map.

In a systematic discussion the extension (differential structure, null cone field) must be contructed in terms of intrinsic structures of $(\tilde{M}, \tilde{g})$.

Traditional method: **employs null geodesics, null hypersurfaces, ...**

Schwarzschild line element $r > 2\,m$

$$\tilde{g} = \left(1 - \frac{2\,m}{r}\right) d\,t^2 - \left(1 - \frac{2\,m}{r}\right)^{-1} d\,r^2 - r^2\,d\,\sigma^2$$

in retarded null coordinate $w = t - (r + 2\,m\,\log(r - 2\,m))$

$$\tilde{g} = \left(1 - \frac{2\,m}{r}\right) dw^2 + 2\,dw\,d\,r - r^2 d\sigma^2.$$

With $\rho = r^{-1}$

$$\Omega = \rho \;\; \textbf{and} \;\; g = \Omega^2\,\tilde{g} = (1 - 2\,m\,\rho)\,\rho^2\,d\,w^2 - 2\,dw\,d\,\rho - \rho^2 d\sigma^2.$$

have smooth extensions to $\mathcal{J}^+ \equiv \{\rho = 0\}$.

Alternative method:

de Sitter space: suitable conformal Gauss gauge based on $\tilde{S} = \{t = 0\}$ gives coordinate transformation $\tau = 2\,\tanh\left(\frac{t}{2}\right)$ and conformal factor $\Theta = 1 - \frac{\tau^2}{4}$.

$$\Theta \;\; \textbf{and} \;\; g = \Theta^2\,\tilde{g} = d\,\tau^2 - (1 + \frac{\tau^2}{4})^2\,d\,\omega^2, \quad \textbf{given on} \;\;]-2, 2[,$$

extend smoothly to the sets $\mathcal{J}^\pm \equiv \{\tau = \pm 2\}$.

Construction of extensions II.

Minkowski space:
suitable conformal Gauss gauge based on $\tilde{S} = \{t = 0\}$ gives a coordinate transformation

$$t = \frac{\frac{\tau}{2}}{\cos^2\frac{\chi}{2} - (\frac{\tau}{2}\,\sin\frac{\chi}{2})^2}, \quad r = \frac{\sin^2\frac{\chi}{2}\,(1 + (\frac{\tau}{2})^2)}{\cos^2\frac{\chi}{2} - (\frac{\tau}{2}\,\sin\frac{\chi}{2})^2},$$

and a conformal factor

$$\Theta = 2\,(\cos^2\frac{\chi}{2} - (\frac{\tau}{2}\,\sin\frac{\chi}{2})^2)$$

on the manifold

$$\tilde{M} = \{0 < \chi < \pi, \quad \vartheta \in S^2, \quad \tau = \pm\sqrt{\frac{1+\cos\chi}{1-\cos\chi}}\}.$$

The fields

$$\Theta \;\; \text{and} \;\; g = \Theta^2\,\tilde{g} = \Theta^2\,(dt^2 - dr^2 - r^2\,d\sigma^2) = \omega^2\,(\frac{1}{\omega^2}\,d\tau^2 - d\chi^2 - \sin^2\chi\,d\sigma^2),$$

with $\omega = 1 + (\frac{\tau}{2})^2$, extend smoothly to

$$\mathcal{J}^{\pm} = \{0 < \chi < \pi, \quad \vartheta \in S^2, \quad \tau = \pm\sqrt{\frac{1+\cos\chi}{1-\cos\chi}}\}.$$

This extension, for which the 1-form b satisfies $< b, \dot{x} > = 0$ on $\{t = 0\}$, does not cover $i^{\pm}$ (conformally invariant statement).
Extensions covering either i^+ or i^- can be constructed by replacing the initial data b by $b + \alpha\,\dot{x}$ with a suitable function α on $\{t = 0\}$.

What happens in the presence of 'strong' curvature ?
Will the curvature necessarily induce the congruence of conformal geodesics to develop caustics ?
Congruences of conformal geodesics can develop caustics which are more severe than those developed by congruences of metric geodesics.

Construction of solutions including their extensions

'There exist smooth (analytic) conformal Gauss systems which cover the complete Schwarzschild-Kruskal space-time and which provide smooth (analytic) conformal extensions to $\mathcal{J}^{\pm}_{1,2}$'.

The underlying congruence of conformal geodesics can be calculated in terms of elliptic integrals.

The regularity of the congruence is shown by deriving and analysing a conformal analogue of the Jacobi equation, which reads in terms of the vacuum-adapted conformal geodesic equations

$$\tilde{\nabla}_X \tilde{\nabla}_X Z = C(X, Z) X + \hat{B}^{\sharp},$$

$$\tilde{\nabla}_X \hat{B} = -\hat{b}\, C(X, Z) + \alpha \tilde{\nabla}_X Z^{\flat} + \gamma X^{\flat},$$

here $\alpha, \gamma = const.$ are along the curves and the tangent vector, the deviation vector field, and the deviation 1-form are denoted by

$$X = \partial_{\bar{\tau}} \bar{x}, \quad Z = \partial_{\lambda} \bar{x}, \quad \hat{B} = \tilde{\nabla}_Z \hat{b}.$$

These observations above suggest:

a systematic way of constructing extensions,

a systematic method of constructing solutions including their extensions: In the three cases above the resulting conformal space-times represent smooth solutions to the general conformal field equations.

Even in the Schwarzschild-Kruskal case the reduced equations reduce to systems or ODE's. On a Schwarzschild part $r > 2m$ the data can be arranged such that the conformal Gauss systems (including their extensions beyond $\mathcal{J}^+$) approach a conformal Gauss system on the (conformally extended) Minkowski space as $m \to 0$.

First issue: existence 'locally near $\mathcal{J}$'.

$\tilde{R}_{\mu\nu} = 0$ **(null infinity light-like):**

'Asymptoytic characteristic initial value problem': data hypersurfaces $\mathcal{N}$, $\mathcal{J}^+_*$ with $\mathcal{N}$ $\sim$ outgoing null hypersurface, $\Sigma = \mathcal{N} \cap \mathcal{J}^+$ space-like 2-surface, $\mathcal{J}^+_*$ $\sim$ part of $\mathcal{J}^+$ in the past of Σ ('the original problem'):

J. Kannar, Proc. Roy Soc. 1996:

For smooth data $\exists_1$ local solution near Σ, i.e. all solutions for which $d^i{}_{jkl}$ has a smooth limit on $\mathcal{J}^+_*$ (and on $\mathcal{N}$) can be characterized.

The freedom to prescribe *null data* on $\mathcal{N}$ and $\mathcal{J}^+_*$ the same as in the characteristic initial value problem for Einstein's equations.

The null data on $\mathcal{J}^+$: *outgoing radiation field*.

$\tilde{R}_{\mu\nu} = \lambda\,\tilde{g}_{\mu\nu}$, **de Sitter-type case $\lambda < 0$, compact time slices (null infinity space-like):**

For smooth data $\exists_1$ local solution near $\mathcal{J}^-$, i.e. all solutions for which $d^i{}_{jkl}$ has a smooth limit on $\mathcal{J}^-$ can be characterized.

With the exception of the mean intrinsic curvature the freedom to specify data on $\mathcal{J}^-$ the same as in a standard Cauchy problem.

Peculiar feature: the Hamiltonian contraint becomes trivial on $\mathcal{J}^-$.

$\tilde{R}_{\mu\nu} = \lambda\,\tilde{g}_{\mu\nu}$, **anti-de Sitter-type case $\lambda > 0$, (null infinity time-like):**

All solutions for which $d^i{}_{jkl}$ has a smooth limit on $\mathcal{J}$ can (locally) be characterized in terms of an initial boundary value problem with (standard) initial data on a space-like slice S and boundary data (3-dim Lorentzian conformal structure) on $\mathcal{J}$: $\exists_1$ local solution near S containing a neighbourhood of $S \cap \mathcal{J}$.

Problem more natural than the standard initial boundary value problem for Einstein's vacuum field equations (cf. H.F., G. Nagy, CMP, 1999).

$\exists$ smooth physical solutions with non-smooth boundary data ??

Second issue: Smooth evolution into $\mathcal{J}^+$.

Properties of $d^i{}_{jkl}$ should be deduced, not postulated. We need to control the behaviour of the solutions as they evolve towards null infinity.

Model case:
$\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu}$, **de Sitter-type case $\lambda < 0$, time slices $\sim S^3$:**

'The asymptotic structure of de Sitter space is non-linearly stable under sufficiently small finite perturbations'.

In our conformal Gauss gauge de Sitter space extends as a solution to the general conformal field equations smoothly into the range $-a \leq \tau \leq a$ for given $a > 2$. The conformal factor becomes negative on the boundary of that domain.

Use the stability properties of the symmetric hyperbolic reduced equations implied by the general conformal field equations to control the perturbed solutions.

In this case null infinity admits a smooth conformal structure and a smooth limit of $d^i{}_{jkl}$ as a consequence of the evolution process and the assumptions on the initial data.

An analogous stability result in the Minkowski-type case is not available. In fact, a result of this simplicity cannot be obtained.