Initial data for stationary space-times near space-like infinity

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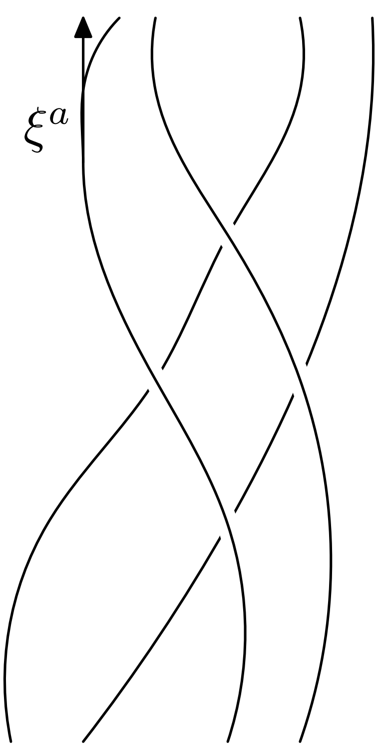
The quotient manifold $\tilde{X}$

ξ^a: timelike Killing vector.

λ, ω : norm and twist of ξ^a.

$(\tilde{M}, \tilde{g}_{ab})$: spacetime.



$\tilde{X}$: collection of all trajectories of ξ^a.

$\tilde{X} = \mathbb{R}^3 \setminus B_R$: B_R is a ball which contains the "sources".

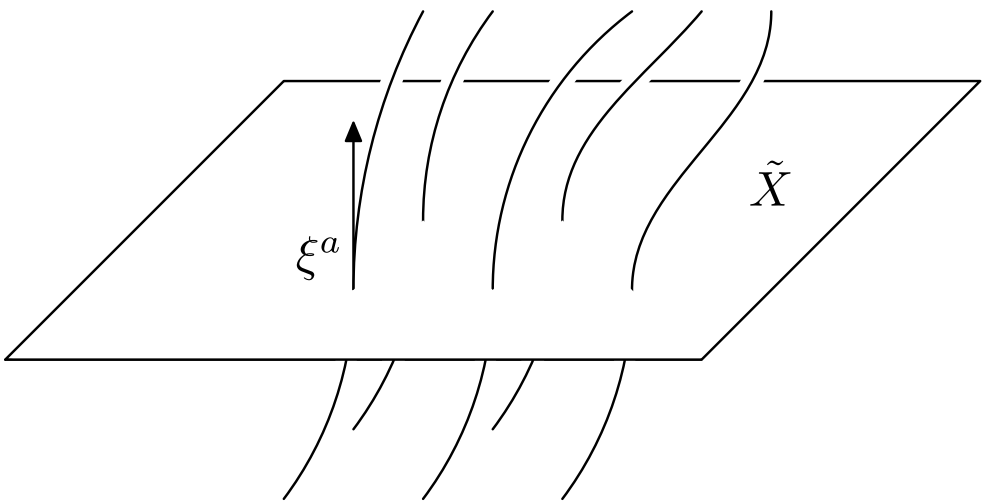
$$\tilde{g}_{ab} = \xi_a \xi_b - \frac{\tilde{\gamma}_{ab}}{\lambda}$$

$\tilde{\gamma}_{ab}$: intrinsic metric on $\tilde{X}$.

Reduction: write the field equations in terms of quantities intrinsic to $\tilde{X}$.

$$(\tilde{M}, \tilde{g}_{ab}) \rightarrow (\tilde{X}, \tilde{\gamma}_{ab})$$

In the static case ($\omega = 0$), $\tilde{X}$ is one of the hypersurfaces of $\tilde{M}$ everywhere orthogonal to ξ^a:



When $\omega \neq 0$, there is no natural way of introducing such surface on $\tilde{M}$.

Stationary spacetimes: initial data

The metric can be written as

$$\tilde{g} = \lambda(dt + \beta_i d\tilde{x}^i) - \lambda^{-1}\tilde{\gamma}_{ij} d\tilde{x}^i d\tilde{x}^j,$$

where λ, β_i, and the Riemannian metric $\tilde{\gamma}_{ij}$ depend only on the spatial coordinates $\tilde{x}^k$.

The intrinsic metric of the hypersurface $\tilde{S}$ defined by $t = constant$ is

$$\tilde{h}_{ij} = \lambda^{-1}\tilde{\gamma}_{ij} - \lambda\beta_i\beta_j,$$

The vector β^k is the essential piece in the translation from the quotient manifold $\tilde{X}$ to the Cauchy slice $\tilde{S}$.

In the static case $\beta^k = 0$ and $\tilde{X} = \tilde{S}$.

Vacuum equations

Hansen potentials:

$$\tilde{\phi}_M = \frac{1}{4\lambda}(\lambda^2 + \omega^2 - 1),$$
$$\tilde{\phi}_S = \frac{1}{2\lambda}\omega,$$
$$\tilde{\phi}_K = \frac{1}{4\lambda}(\lambda^2 + \omega^2 + 1).$$

These functions satisfy the relation

$$\tilde{\phi}_M^2 + \tilde{\phi}_S^2 - \tilde{\phi}_K^2 = -\frac{1}{4}.$$

Denote by $\tilde{\phi}$ any of the functions $\tilde{\phi}_M, \tilde{\phi}_S, \tilde{\phi}_K$.

The unknowns are $\tilde{\gamma}_{ab}, \tilde{\phi}_M, \tilde{\phi}_S$.

The vacuum field equations are:

$$\tilde{\Delta}\tilde{\phi} = 2\tilde{R}\tilde{\phi}$$
$$\tilde{R}_{ij} = 2(\tilde{D}_i\tilde{\phi}_M\tilde{D}_j\tilde{\phi}_M + \tilde{D}_i\tilde{\phi}_S\tilde{D}_j\tilde{\phi}_S - \tilde{D}_i\tilde{\phi}_K\tilde{D}_j\tilde{\phi}_K),$$

where $\tilde{R}_{ij}$ is the Ricci tensor of $\tilde{\gamma}_{ij}$ and $\Delta = \tilde{D}^i\tilde{D}_i$.

Conformal compactification of $\tilde{X}$

We assume that $(\tilde{X}, \tilde{\gamma}_{ij}, \tilde{\phi}_M, \tilde{\phi}_S)$ is asymptotically flat in the following sense. There exists a manifold X, consisting of $\tilde{X}$ and an additional point i; such that:

(i) For some real constant $B^2 > 0$ the conformal factor

$$\Omega = \frac{1}{2} B^{-2} [(1 + 4(\tilde{\phi}_M^2 + \tilde{\phi}_S^2))^{1/2} - 1]$$

is C^2 on X and satisfies

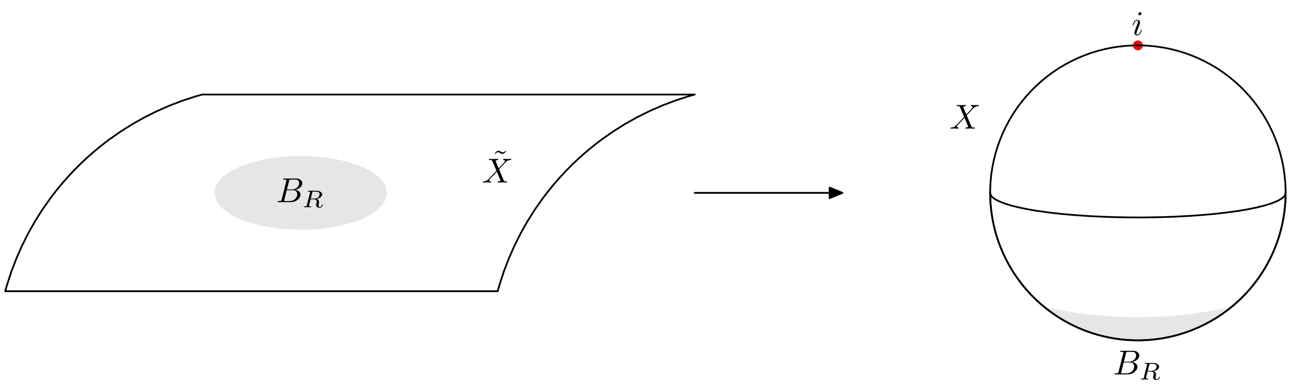
$$\Omega(i) = 0, \quad D_i \Omega(i) = 0$$

at the point i.

(ii) $\gamma_{ij} = \Omega^2 \tilde{\gamma}_{ij}$ extends to a $C^{4,\alpha}$ metric on X and

$$D_j D_k \Omega(i) = 2\gamma_{jk}(i).$$

(iii) Ω is $C^{2,\alpha}$ on X.

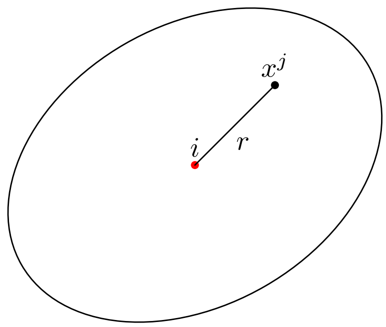

B_R
$\tilde{X}$
i
X
B_R

Define the rescaled functions $\phi = \tilde{\phi}/\sqrt{\Omega}$.

It is possible to find an elliptic system of equations for the rescaled quantities $\gamma_{ab}, \phi_M, \phi_S$ which is regular at i.

Theorem 1 (Beig-Simon) *For any asymptotically flat solution $(\tilde{\gamma}_{ij}, \tilde{\phi}_M, \tilde{\phi}_S)$ of the stationary, vacuum, field equations there exists a chart defined in some neighborhood of i in X such that $(\gamma_{ij}, \phi_M, \phi_S, \Omega)$ are analytic.*

$(\gamma_{ij}, \phi_M, \phi_S)$ are the "natural" fields on X.



x^j: γ-normal coordinates x^j centered at i.

$\gamma_{ij}, \phi_M, \phi_S$: are analytic with respect to x^j.

$r = (\sum_{j=1}^{3} (x^j)^2)^{1/2}$: radial coordinate.

r, $\sqrt{\Omega}$: are not analytic at i.

λ, ω, ϕ_K and β^i are not analytic at i:

$$\phi_K = \frac{f_K}{\sqrt{\Omega}}, \quad f_K = \sqrt{\Omega(\phi_M^2 + \phi_S^2) + 1/4},$$

where f_K is analytic;

$$\omega = \frac{\phi_S}{(\phi_K - \phi_M)}, \quad \lambda = \frac{1}{2\sqrt{\Omega}(\phi_K - \phi_M)}.$$

$$\Omega = r^2 f_\Omega,$$

where f_Ω is analytic and $f_\Omega(i) = 1$.

The non-analytic type of behavior of these fields is given by the radial coordinate r.

Conformal compactification of the initial data $\tilde{S}$

The conformal metric on S is given by:

$$h_{ij} = \gamma_{ij} - \Omega^2 \lambda^2 \beta_i \beta_j.$$

How differentiable is h_{ab} at i?

The vector β^i

β^i satisfies the following equation

$$\tilde{D}_i\omega = -\lambda^2\tilde{\epsilon}_{ijk}\tilde{D}^j\beta^k.$$

We have the freedom

$$\beta_k \rightarrow \beta_k + \partial_k f,$$

which correspond to a change in the foliation

$$t \rightarrow t - f.$$

Lemma 1 *There exists a solution β_i which, in normal coordinates x^i, has the following form*

$$\beta_i = \beta_i^1 + \frac{\beta_i^2}{r},$$

where β_i^1 and β_i^2 are analytic functions of x^i given by

$$\beta_i^1 = \epsilon_{ijk} f_1^j x^k, \quad \beta_i^2 = \epsilon_{ijk} f_2^j x^k,$$

where f_1^i and f_2^i are analytic. In particular, this implies that $\beta_i x^i = 0$.

Theorem 2 *Let β^i be given by the previous lemma. In some neighborhood of i, the metric h_{ij} has the form*

$$h_{ij} = h_{ij}^1 + r^3 h_{ij}^2,$$

where h_{ij}^1 and h_{ij}^2 are analytic.

Remarks:

i) $h_{ij} \in W^{4,p}$, $p < 3$, because $r^3 \in W^{4,p}$, $p < 3$. It implies, in particular, that the metric is in $C^{2,\alpha}$.

ii) The metric is analytic in the spherical coordinates r, θ, φ.

Application: stationary spacetimes have analytic scri

$$\tilde{g} = \lambda(dt + \beta_i d\tilde{x}^i) - \lambda^{-1}\tilde{\gamma}_{ij} d\tilde{x}^i d\tilde{x}^j,$$

where λ, β_i, and $\tilde{\gamma}_{ij}$ are analytic functions of the spherical coordinates $1/\tilde{r}$, θ, φ. Using this we can prove the following:

Theorem 3 (Damour-Schmidt) *Every stationary, asymptotically flat, vacuum spacetime admits an analytic conformal extension through null infinity.*