Cheeger-Gromov Theory and Applications to General Relativity

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1^{st} Level Problem

Bound local geometry in terms of

$$|\mathbf{R}|_{L^\infty} \leq K. \tag{5.1}$$

Usual norm of curvature tensor

$$|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \text{ not } \geq 0. \tag{5.2}$$

For Ricci-flat Lorentz metrics, 2 scalar invariants of full curvature

$$|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \text{ and } <\mathbf{R}, *\mathbf{R}>= \mathbf{R}_{ijkl}(*\mathbf{R}^{ijkl}).$$

Both can vanish on non-flat space-times: e.g.

plane-fronted gravitational waves

$$\mathbf{g} = -2dudv + 2(dx^2 + dy^2) - 2h(u, x, y)du^2,$$

$$\Delta_{(x,y)}h = 0, \ h \text{ arbitrary in } u.$$

Class of such highly non-compact $\Rightarrow$ no local control of metric in any coord. system, under bounds (5.2).

Size conditions. Let Ω = domain in a smooth Lorentz manifold $(\mathbf{M},\ \mathbf{g})$, with smooth time function $T = \partial/\partial t$. Let $S = t^{-1}(0)$ and suppose the 1-cylinder

$$C_1 = B_p(1) \times [-1, 1] \subset\subset \Omega.$$

Let $D = ImT|_{C_1} \subset\subset T^+\Omega$.

Theorem 5.1 *Suppose Ω satisfies the size conditions and $\exists$ constants $K < \infty$, $v_o > 0$ s.t.*

$$|\mathbf{R}|_T \leq K, \quad vol_g B_p(1) \geq v_o. \tag{5.4}$$

Then $\exists r_o > 0$, $R_o < \infty$, depending only on K, v_o and D, and coord. charts on the r_o-cylinder

$$C_{r_o} = B_p(r_o) \times [r_o, r_o] \subset C_1,$$

s.t. on C_{r_o},

$$||\mathbf{g}_{\alpha\beta}||_{L^{2,p}} \leq R_o. \tag{5.5}$$

2^{nd} Level Problem

Replace $|\mathbf{R}|_T$ bound by bound on $|Ric_{\mathbf{g}}|$.

For vacuum space-times: remove bound on $|\mathbf{R}|_T$.

Seek analogues of Convergence I, II results:

bound on Ric, inj/1-cross $\Rightarrow L^{2,p}$ control on g

Two parts to proof: geometric/analytic

geometric

• Splitting theorem
Have direct analogue.

Lorentzian Splitting Theorem (Eschenburg, Galloway, Newman) $(\mathbf{M},\ \mathbf{g})$ time-like geodesically complete or globally hyperbolic vacuum space-time which contains a time-like line, then $(\mathbf{M},\ \mathbf{g})$ is flat.

Can define Lorentzian 1-cross: $|T|^2 = -1$

$$Cro_1(x,T) = \sup\{t : \gamma = \text{max. geod. on}[-t,t], \gamma(0) = x, \gamma'(0) = T\},$$

$$Cro_1(\Omega,T) = \inf_{x\in\Omega} Cro_1(x,T).$$

analytic

Missing step - No regularity boost from hyperbolic PDE.

However, smoothness of initial data preserved until hit boundary of maximal development.

Let $S \subset (\mathbf{M}, \mathbf{g})$ = space-like hypersurface. Define H^s harmonic radius $\rho_s(x)$, $x \in S$, $s > 2.5$, (large) as before: largest radius s.t.

$$[\rho_s(x)]^{2s-3} \int_{B_x(\rho_s(x))} |\partial^s g_{\alpha\beta}|^2 \leq C.$$

Suppose S = hypersurface with smooth, (C^∞), initial data. Let S_t = hypersurface obtained from vacuum evolution. Then (Choquet-Bruhat)

$$\min_{x_t \in S_t} \rho_s(x_t) \geq c_1 \Rightarrow \min_{x_t \in S_t} \rho_{s+1}(x_t) \geq c_2, \tag{5.6}$$

where c_2 depends on c_1 and the initial data set.

We raise the following:

Regularity Problem. Can the estimate (5.6) be improved to an estimate

$$\min_{x_t \in S_t} \rho_{s+1}(x_t) \geq c_0 \min_{x_t \in S_t} \rho_s(x_t), \tag{5.7}$$

where c_0 depends only on the initial data set?

The important point of (5.7) over (5.6) is that the estimate (5.7) is scale-invariant.

If (5.7) holds, it serves as an analogue of the regularity boost. Can then imitate the proofs of Riemannian convergence results to obtain Lorentzian convergence.

Would have numerous interesting applications.

Next, drop any assumption on the 1-cross of $(\mathbf{M},\ \mathbf{g})$: maintain only a lower bound on the volumes of geodesic balls on space-like hypersurfaces.

This leads to issues of singularity formation and the structure of the boundary of the vacuum space-time; little understood mathematically.

Sandwich Problem: Version I.

Let $(\mathbf{M}, \mathbf{g}_i)$ be a sequence of vacuum space-times, and let Σ_i^1, Σ_i^2 be two compact Cauchy surfaces in $\mathbf{M}$, with $\Sigma_i^2 \gg \Sigma_i^1$ and with

$$1 \leq dist_{\mathbf{g}}(x, \Sigma_i^1) \leq 10, \quad \forall x \in \Sigma_i^2.$$

Suppose the Cauchy data (g_i^j, K_i^j), $j = 1, 2$ on each Cauchy surface are uniformly bounded in H^s, for some fixed $s > 2.5$, possibly large. Hence the data (g_i^j, K_i^j) converge, in a subsequence and weakly in H^s, to limit H^s Cauchy data g_∞^j, K_∞^j on Σ^j.

Do the vacuum space-times $A_i(1,2) \subset (M, g_i)$ between Σ^1 and Σ^2 converge, weakly in H^s, to a limit space time,

$$(A_i(1,2), g_i) \to (A_\infty, g_\infty)? \tag{5.8}$$

Application. Understand limits of the AS vacuum perturbations of deSitter, (Friedrich).

The sandwich problem asks: suppose one has control on the space-time near past and future space-like infinity $\mathcal{I}^{\pm}$, does it follow that one has control in between?

Sandwich Problem: Version II. ($\mathbf{M}$, $\mathbf{g}$) smooth vacuum space-time, Σ^1, Σ^2 smooth compact space-like hypersurfaces,

$$\Sigma^1 \gg \Sigma^2.$$

Does the maximal globally hyperbolic development from Σ^1 contain Σ^2?

Can ($\mathbf{M}$, $\mathbf{g}$) have an "invisible" singularity in between?

§6. Future Asymptotics and Geometrization of 3-Manifolds

Issue: Understand future asymptotics of vacuum cosmological space-times $(\mathbf{M},\ \mathbf{g})$.

$(\mathbf{M},\ \mathbf{g})$ contains compact CMC Cauchy surface Σ,

$$\sigma(\Sigma) \leq 0. \tag{6.1}$$

$\exists$ foliation $\mathcal{F}$ by CMC Cauchy surfaces $\Sigma_\tau \approx \Sigma$,

$$\tau = \text{mean curvature } \in (-\infty, 0). \tag{6.2}$$

$\tau \uparrow$ to future – expanding direction. Let $\mathbf{M}_{\mathcal{F}}$ = foliated region in $\mathbf{M}$.

Suppose $(\mathbf{M},\ \mathbf{g})$ geodesically complete to future of Σ and, to future of Σ,

$$\mathbf{M} = \mathbf{M}_{\mathcal{F}}.$$

Strong assumptions, but necessary to study smooth asymptotics.

Induced metrics $g_\tau = \mathbf{g}|_\Sigma$ = curve of Riemannian metrics on fixed 3-manifold Σ. As $\tau \to 0$, typically

$$vol_{g_\tau}\Sigma \to \infty, \ \ g_\tau \text{ becomes flat}$$

– due to expansion.

Not so interesting. To study asymptotics, rescale by distance tp fixed base point – "blow-down".

For $x \gg \Sigma$, let

$$t(x) = dist_{\mathbf{g}}(x, \Sigma)$$

and

$$t_\tau = t_{max}(\tau) = max\{t(x) : x \in \Sigma_\tau\}. \tag{6.3}$$

Thus study the asymptotic behavior of the metrics

$$\bar{g}_\tau = t_\tau^{-2} g_\tau, \tag{6.4}$$

on Σ_τ. For rescaled space-time $(\mathbf{M}, \bar{\mathbf{g}}_\tau)$, distance of $(\Sigma_\tau, \bar{g}_\tau)$ to "big bang" $\to 1$, as $\tau \to 0$.

Definition 6.1 Σ a closed, oriented, connected 3-manifold, non-positive Yamabe type. A *weak* geometrization of Σ is a decomposition

$$\Sigma = H \cup G : \tag{6.5}$$

• H = finite collection of complete, connected hyperbolic manifolds, of finite volume $\subset \Sigma$.
• G = finite collection of connected graph manifolds $\subset \Sigma$.
• Union along a finite collection of tori $\mathcal{T} = \cup T_i = \partial H = \partial G \subset \Sigma$.

A *strong* geometrization of Σ is a weak geometrization s.t.

$$\pi_1(T^2) \hookrightarrow \pi_1(\Sigma), \ \forall T^2 \in \mathcal{T}.$$

$\mathcal{T} = \emptyset \Rightarrow$ weak = strong.

$\exists$ sequences of metrics g_i which limit on a geometrization of Σ.
• $g_i \to$ hyperbolic metric on H
• $g_i \to$ collapse on G
match behaviors far down hyperbolic cusps.

Curvature assumption. Assume $\exists C < \infty$ s.t. for $x \gg \Sigma$,

$$|\mathbf{R}|(x) + t(x)|\nabla\mathbf{R}|(x) \leq C \cdot t^{-2}(x). \tag{6.6}$$

Curvature norm $|\mathbf{R}| = |\mathbf{R}|_T$, T = unit normal to the foliation Σ_τ. Bound (6.6) scale-invariant.

- holds for Bianchi space-times
- Conjecture: holds for perturbations of Bianchi space-times
- probably holds for Gowdy

No known cosmological vacuum space-times, geodesically complete to future, where it fails.

Theorem 6.2 *Let* ($\mathbf{M}$, $\mathbf{g}$) *be a cosmological space-time of non-positive Yamabe type. Suppose that the curvature assumption (6.6) holds, and that* $M_{\mathcal{F}} = \mathbf{M}$.

Then ($\mathbf{M}$, $\mathbf{g}$) *is future geodesically complete and, for any sequence* $\tau_i \to 0$*, the slices* $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$ *have a subsequence converging to a weak geometrization of* Σ.

Ideas in Proof: Curve of metrics $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$, blown-down.

- L^∞ bound on $|\mathbf{R}|_T \Rightarrow L^\infty$ bound on intrinsic and extrinsic curvature of slices $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$. Proof similar to Theorem 5.1.
- Can apply Cheeger-Gromov theory: subsequences either converge, collapse or form cusps.

Collapse/cusps $\Rightarrow$ graph manifold structure.

Remains to show that convergence/thick part converges always to hyperbolic manifolds.

Main ingredient for convergence: Volume monotonicity:

$$\frac{vol_{g_\tau}\Sigma_\tau}{t_\tau^3}\downarrow, \tag{6.7}$$

Analogous to monotonicity of reduced Hamiltonian (Fischer-Moncrief).

This monotonicity follows from simple analysis of the Raychaudhuri equation, (as in Penrose-Hawking singularity theorems), together with a suitable maximum principle.

Further, the ratio in (6.7) is constant on some interval $[\tau_1, \tau_2]$ iff the annular region $A(\tau_1, \tau_2) = \tau^{-1}(\tau_1, \tau_2)$ = annulus in a *flat* Lorentzian cone

$$\mathbf{g_o} = -dt^2 + t^2 g_{-1},$$

where g_{-1} is a hyperbolic metric.

Again, the ratio in (6.7) is scale invariant, and so

$$\frac{vol_{g_\tau}\Sigma_\tau}{t_\tau^3} = vol_{\bar{g}_\tau}\Sigma_\tau.$$

Non-collapse means this volume is bounded below. Hence, it converges to a non-zero limit, so have equality in limit, so space-time limit is flat Lorentz cone.