Applications

- Re: the null splitting theorem: *If M is null geodesically complete, obeys the null energy condition, and contains a null line η then η is contained a smooth closed achronal totally geodesic null hypersurface.*
- Re: the rigidity philosophy.
- Re: the proof. Use the max prin for C^0 null hypersurfaces to show that

$$S_+ = \partial I^+(\eta) \quad \text{and} \quad S_- = \partial I^-(\eta)$$

  agree, and form a smooth totally geodesic null hypersurface.
- Re: the completeness assumption. Proof only requires null generators of S_+ to be past complete, and null generators of S_- to be future complete.
- NEC and the Einstein equations: If the Einstein equations with cosmological constant hold,

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = 8\pi T_{ij}$$

  then

$$\mathrm{Ric}\,(X, X) = 8\pi T_{ij} X^i X^j$$

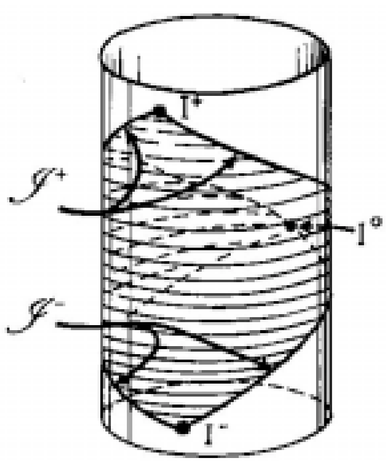
  for all null vectors X.

  Hence the NEC is insensitive to the sign of the cosmological constant. In particular in the vacuum case, $T_{ij} = 0$, the NEC is always satisfied, regardless of the sign of Λ.

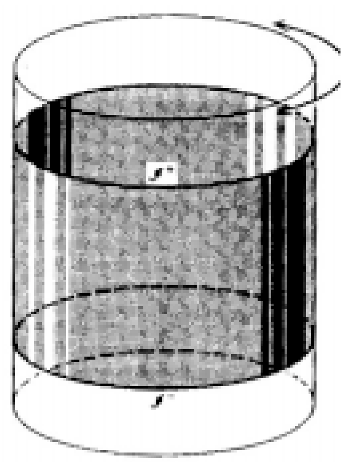
Consider now some global results for solutions to the Einstein equations with prescribed asymptotics.

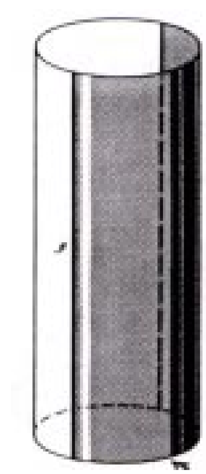
Use Penrose's notion of conformal infinity to treat the asymptotics.

Based on the conformal imbeddings of Minkowski space, de Sitter space and anti-de Sitter space into the Einstein static universe:



Minkowki


de Sitter


anti-de Sitter

We are going to focus primarily on spacetimes which are asymptotically de Sitter, i.e., for which the conformal boundary $\mathcal{J}$ is spacelike.

Def. (M, g) is a spacetime of *de Sitter type* provided there exists a *smooth* spacetime-with-boundary $(\tilde{M}, \tilde{g})$ and a *smooth* function Ω on $\tilde{M}$ such that

- M is the interior of $\tilde{M}$; hence $\tilde{M} = M \cup \mathcal{J}$, $\mathcal{J} = \partial\tilde{M}$.
- $\tilde{g} = \Omega^2 g$, where (i) $\Omega > 0$ on M, and (ii) $\Omega = 0$, $d\Omega \neq 0$ along $\mathcal{J}$.
- $\mathcal{J}$ is spacelike.

$\mathcal{J}$ decomposes into two disjoint sets,

$$\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$$

where, $\mathcal{J}^+ \subset I^+(M, \tilde{M})$ and $\mathcal{J}^- \subset I^-(M, \tilde{M})$.

Def. A spacetime M of de Sitter type is *asymptotically simple* provided each inextendible null geodesic in M has a future end point on $\mathcal{J}^+$ and a past end point on $\mathcal{J}^-$.

Ex. De Sitter space, which can be expressed in global coordinates as,

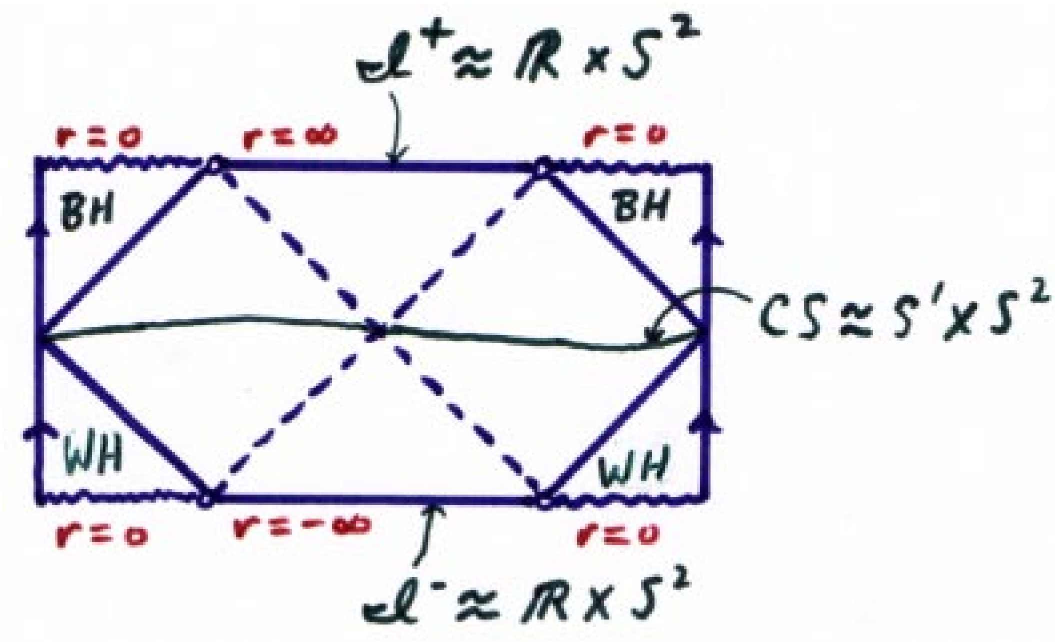
$$M = \mathbb{R} \times S^n, \qquad ds^2 = -dt^2 + \cosh^2 t\, d\Omega^2$$

Ex. Schwarzschild-de Sitter space (dim 4).

$$ds^2 = -(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)\, dt^2 + (1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2)^{-1} dr^2 + r^2 d\omega^2\,,$$

where $\Lambda > 0$ (and $9\Lambda m^2 < 1$).

Penrose diagram:



SS-DeS is a spacetime of de Sitter type, but is not asymptotically simple.

<u>*Ex.*</u> FRW spacetime,

$$M = \mathbb{R} \times \Sigma \,, \qquad ds^2 = -dt^2 + a^2(t) d\sigma^2$$

which is a solution to the Einstein equations with perfect fluid source and $\Lambda > 0$.

Starts from a big bang but behaves like de Sitter to the far future.

For such models, $\mathcal{J} = \mathcal{J}^+$, i.e., there is a future conformal infinity, but no past conformal infinity. Shall also refer to such spacetimes as being of de Sitter type.

Asymptotic simplicity can be related to the causal structure of of spacetime.

Prop. *Let M be a spacetime of de Sitter type with future conformal infinity $\mathcal{J}^+$.*

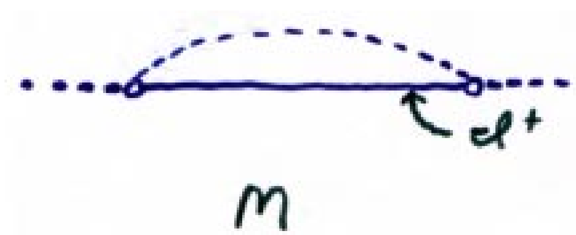
(1) If M is future asymptotically simple then M is globally hyperbolic.

(2) If M is globally hyperbolic and $\mathcal{J}^+$ is compact then M is future asymptotically simple.

In either case, the Cauchy surfaces of M are homeomorphic to $\mathcal{J}^+$.

Comments on proof.

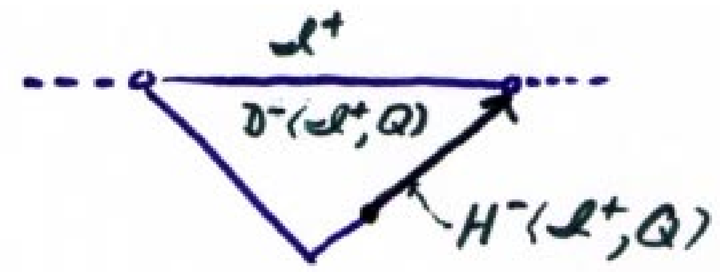
(1): Extend $M \cup \mathcal{J}^+$ a little beyond $\mathcal{J}^+$ to obtain a spacetime without boundary Q such that $\mathcal{J}^+$ is a *future* Cauchy surface in Q:



i.e. such that $D^+(\mathcal{J}^+, Q) = J^+(\mathcal{J}^+, Q) \iff H^+(\mathcal{J}^+, Q) = \emptyset$.

We claim $H^-(\mathcal{J}^+, Q) = \emptyset$, as well, and hence $\mathcal{J}^+$ is a Cauchy surface for Q.

Suppose $H^{-}(\mathcal{J}^{+}, Q) \neq \emptyset$:



By asymptotic simplicity, null generators of $H^{-}(\mathcal{J}^{+}, Q)$ must meet $\mathcal{J}^{+}$ $\rightarrow\leftarrow$.

Thus $\mathcal{J}^{+}$ is Cauchy for Q, and Q is globally hyperbolic. One can then construct a Cauchy surface for Q lying entirely in M. This is easily seen to be a Cauchy surface for M, as well, and hence M is globally hyperbolic.

Finally, since all Cauchy surfaces are homeomorphic, the Cauchy surfaces of M are homeomorphic to $\mathcal{J}^{+}$.

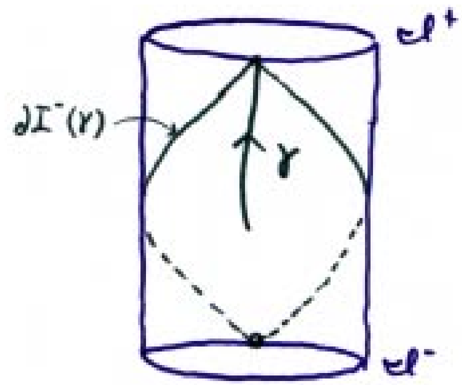
(2): Similar arguments involved. Uses the basic fact:

Prop. If S is a compact achronal hypersurface in a globally hyperbolic spacetime M then S must be a Cauchy surface for M.

Uniqueness results for spacetimes of de Sitter type.

Every null geodesic in de Sitter space is a null line (inextendible achronal null geodesic).

This is related to the fact that the observer horizon $\partial I^-(\gamma)$ of every observer (future inextendible timelike curve) γ is eternal, i.e. extends from $\mathcal{J}^+$ to $\mathcal{J}^-$.



Theorem. *Suppose M^4 is an asymptotically simple spacetime of de Sitter type satisying the vacuum Einstein equation,*

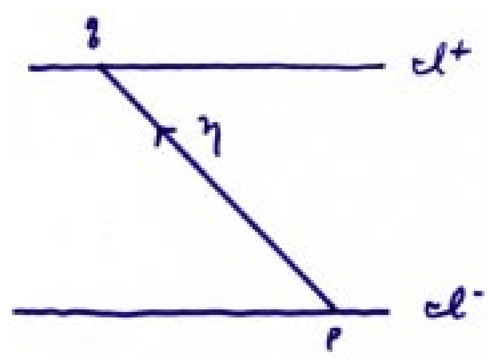
$$\mathrm{Ric} = \lambda g$$

with $\lambda > 0$. If M contains a null line then M is isometric to de Sitter space.

Comment: This can be interpreted in terms of the initial value problem, due to Friedrich's results on the nonlinear stability of asymptotic simplicity, in the case $\Lambda > 0$.

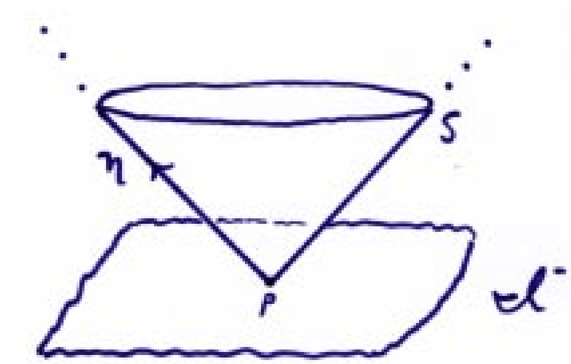
- In general, a small perturbation of the initial data in de Sitter space destroys *all* the null lines, i.e.,
- in the perturbed spacetime, there are no eternal observer horizons.

Proof. The main step is to show M has constant curvature.



η is contained in a smooth totally geodesic null hypersurface S. Focus on situation near p:

$$S = \partial I^+(\eta, M) = \partial I^+(p, \tilde{M}) \cap M$$



Thus, $N_p = S \cup \{p\}$ is a smooth null cone in $\tilde{M}$. Since the shear σ is a conformal invariant, the null generators of N_p have vanishing shear.

The trace free part of the Ricatti equation then implies,

$$\tilde{C}_{aKbK} = 0 \qquad (\iff \tilde{\psi}_0 = 0)$$

By an argument of Friedrich '86,

$$C^i_{jkl} = 0 \qquad \text{on } D^+(N_p, \tilde{M}) \cap M$$

Argument makes use of the conformal field equations, specifically,

$$\tilde{\nabla}_i d^i_{jkl} = 0, \qquad d^i_{jkl} = \Omega^{-1} C^i_{jkl}$$

Time-dually, C^i_{jkl} vanishes on $D^-(N_q, \tilde{M}) \cap M$, and hence on all of M.

Thus M has constant curvature. It can be further shown that M is geodesically complete and simply conncted, and so M is isometric to de Sitter space.

<u>*Comments.*</u>

(1) The assumption of asymptotic simplicity cannot be removed, cf., SS-deS space. But it appears it can be substantially weakened.

¿**Theorem**? *Suppose M is a maximally globally hyperbolic spacetime of de Sitter type satisying the vacuum Einstein equation,*

$$\mathrm{Ric} = \lambda g$$

with $\lambda > 0$. If M contains a null line with end points on $\mathcal{J}$ then M is isometric to de Sitter space.

(2) Analogous result holds for Minkowski space.

Theorem. *Suppose M^4 is an asymptotically simple spacetime satisying the vacuum Einstein equation,*

$$\mathrm{Ric} = 0\,.$$

If M contains a null line then M is isometric to Minkowski space.

Remarks:

- Due to Corvino-Chrusćiel-Delay, this result is not vacuous!
- Asymptotic simplicity assumption is not so easily weakened in this case.
- This result should continue to hold in the nonvacuum case for certain fields (matter fields, EM, Yang-Mills).

Results on the topology of spacetimes of de Sitter type.

Q. What are the allowable spatial topologies within the class of asymptotically simple and de Sitter solutions of the Einstein equations?

Theorem (Andersson, G.) *Let M^{n+1}, $n \geq 2$, be a spacetime of de Sitter type with past and future conformal boundaries $\mathcal{J}^{\pm}$. Assume that M is asymptotically simple either to the past or future. Assume further that M obeys the null energy condition.*

Then M is globally hyperbolic, and the Cauchy surfaces for M are compact with finite fundamental group.

Comments.

(1) Thus, in $3+1$, the Cauchy surfaces are homotopy 3-spheres, perhaps with identifications.

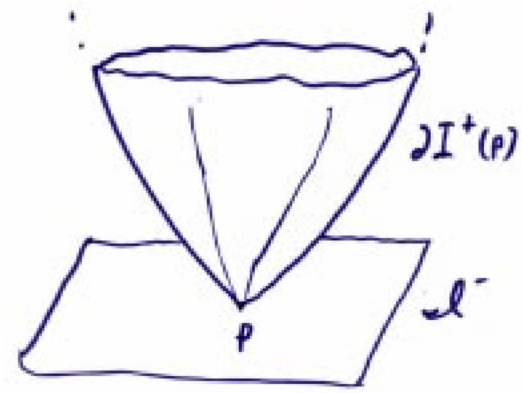
(2) In particular, the Cauchy surfaces cannot have topology $S^2 \times S^1$. Or, put another way if the Cauchy surface topology is $S^2 \times S^1$, then M cannot be asymptotically simple, either to the future or the past; cf., SS-deS.

Proof. We show the Cauchy surfaces of M are compact.

Can extend M a little beyond $\mathcal{J}^{\pm}$ to obtain a spacetime $P \supset \tilde{M}$ such that any Cauchy surface for M is a Cauchy surface for P.

Suffices to show the Cauchy surfaces of P are compact.

Fix $p \in \mathcal{J}^-$, and consider $\partial I^+(p, P)$:

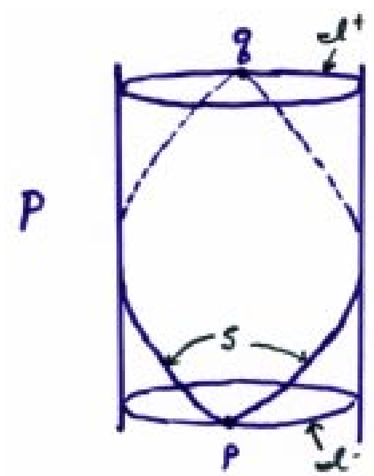


If $\partial I^+(p, P)$ is compact then $\partial I^+(p, P)$ is a compact CS for P and we are done.

If $\partial I^+(p, P)$ is noncompact then can construct null geodesic generator $\gamma \subset \partial I^+(p, P)$ which is future inextendible in P.

By future asymptotic simplicity, γ meets $\mathcal{J}^+$ at q, say. γ_0, the portion of γ from p to q is a null line in M.

By the null splitting theorem, γ_0 is contained in a totally geodesic null hypersurface S. By previous arguments, $N = S \cup \{p, q\}$ is a compact achronal hypersurface in P:



Hence, N is a compact Cauchy surface for P. Thus the Cauchy surfaces for M are compact.

We consider a related result which is an application of the *Penrose singularity theorem*,

Theorem (Penrose). *Let M be a globally hyperbolic satisfying the null energy condition. Then the following conditions* cannot *all hold.*

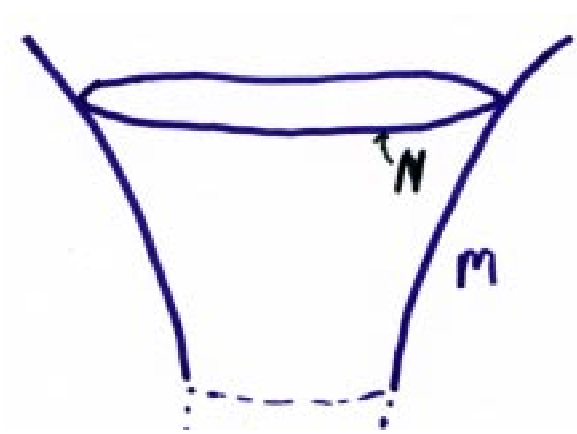
- *The Cauchy surfaces of M are non-compact.*
- *M contains a past-trapped surface.*
- *M is past null geodesically complete.*

Theorem (Andersson, G.) *Suppose M^{n+1}, $2 \leq n \leq 7$, is a globally hyperbolic spacetime of de Sitter type, with future conformal boundary $\mathcal{J}^+$, which is compact and orientable. Suppose further that M obeys the null energy condition.*

If the Cauchy surfaces of M have positive first Betti number, $b_1 > 0$, then M is past null geodesically incomplete.

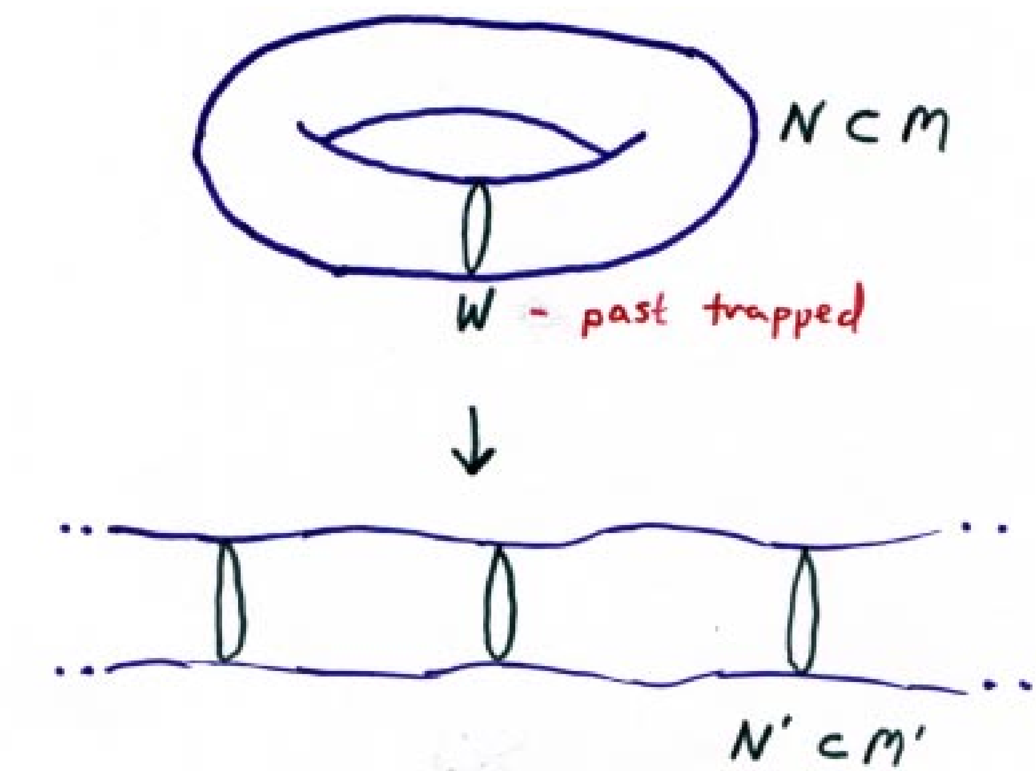
Discussion of proof.

In the far future, can choose a CS N for M, with second fundamental form which is positive definite wrt the *future* pointing normal.



Now, $b_1(N) > 0 \iff H_{n-1}(N, \mathbb{Z}) \neq 0$.

Minimizing area in homology class, obtain a homologically nontrivial smooth compact orientable minimal hypersurface surface $W \subset N$:



The preimage of W in the covering spacetime consists of infinitely many copies of W each past trapped, contained in a noncompact CS. Thus M', and hence M must be past null geodesically complete.

<u>*Comment:*</u> Further results on the topology of spacetimes of de Sitter type may be found in: Andersson and G., hep-th/0202161.