Classification of Prime 3-Manifolds with Yamabe Invariant Greater Than $\mathbb{RP}^3$

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(joint work with André Neves)

(M^3, g)

Define the Yamabe energy functional of (M^3, g) to be the average value of the scalar curvature R_g of (M^3, g) after g has been scaled to have total volume 1. More explicitly,

$$E(g) = \frac{\int_M R_g \, dV_g}{\left(\int_M dV_g\right)^{1/3}}.$$

Note that on S^3,

(S^3, g_0)

$$E(g_0) = 6 \cdot (2\pi^2)^{2/3} \equiv Y_1.$$

$R_0 \equiv 6$

$\mathrm{Vol}(g_0) = 2\pi^2$

Define the Conformal Yamabe energy

$$C(g) = \inf \left\{ E(\bar{g}) \mid \bar{g} = u(x)^4 g ,\ u > 0,\ u \in H^1 \right\}$$

where we note that

$$E(\bar{g}) = \frac{\int_M \left(8 |\nabla u|_g^2 + R_g u^2 \right) dV_g}{\left(\int_M u^6 \, dV_g \right)^{1/3}} .$$

Note also that on any conformal class of any manifold, $u(x)$ may be concentrated such that

$(M^3, \bar{g})$ (S^3, g_0)

$E\Big($  $\Big) \approx E\Big($  $\Big) = Y_1$,

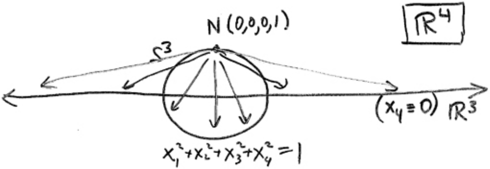
so

$$C(g) \leq Y_1 .$$

Thus we may define

$$Y(M) = \sup_g C(g) \leq Y_1 .$$

Stereographic Projection



Exercise:

Show that the projection map from $S^3 - \{N\}$ to $\mathbb{R}^3$ is conformal.

Answer: The metric on $S^3 - \{N\}$ using this $\mathbb{R}^3$ as a coordinate chart is

$$g_{ij} = \left(\frac{2}{1+|x|^2}\right)^2 \delta_{ij} .$$

A closed 3-manifold M^3 is prime if

$M^3 = A \# B$ implies that either A or B is S^3.

3-manifolds always have a unique prime factorization modulo the relation

$$(S^2 \times S^1) \# (S^2 \tilde{\times} S^1) = (S^2 \tilde{\times} S^1) \# (S^2 \tilde{\times} S^1)$$

where $S^2 \tilde{\times} S^1$ is the nonorientable S^2 bundle over S^1. Thus, it makes sense to try to classify prime 3-manifolds. One approach is to list them in order of their Yamabe invariants.

Theorem 1: The first five prime 3-manifolds ordered by their Yamabe invariants are

$$S^3,\ S^2 \times S^1,\ S^2 \tilde{\times} S^1,\ \mathbb{RP}^3, \text{ and } \mathbb{RP}^2 \times S^1.$$

This theorem will follow from our main result:

Theorem 2: A closed 3-manifold with $Y > Y_2 = \frac{Y_1}{2^{2/3}}$ is either S^3 or a connect sum with an S^2 bundle over S^1.

Corollary 1: $Y(\mathbb{RP}^3) = Y_2 = \frac{Y_1}{2^{2/3}}$.

Pf.: $c($  $) = Y_2$.

$(\mathbb{RP}^3, g_0)$

Corollary 2: $Y(\mathbb{RP}^2 \times S^1) = Y_2$.

Pf.: $c($  $) \to Y_2$

$\mathbb{RP}^2 \times S^1$

Corollary 3:

$$Y(\underbrace{\mathbb{RP}^3 \# \mathbb{RP}^3 \# \dots \# \mathbb{RP}^3}_{n \text{ times}} \# \overbrace{(\mathbb{RP}^2 \times S^1) \# \dots \# (\mathbb{RP}^2 \times S^1)}^{m \text{ times}}) = Y_2$$

for any $m, n \geq 1$.

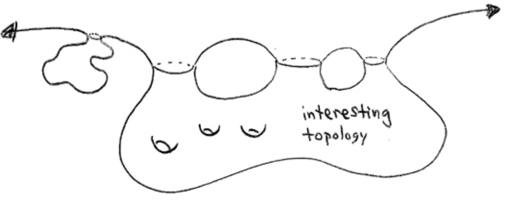
Pf: O. Kobayashi's connect sum estimate.

Corollary 4: The only closed, simply connected 3-manifold with $Y > Y_2$ is S^3.

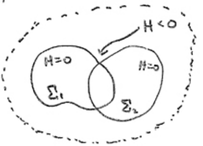
heorem (Meeks - Simon - Yau)

he exterior of the outermost minimal surface of an symptotically flat 3-manifold is diffeomorphic to

$$\mathbb{R}^3 \setminus \bigcup_{\text{finite}} \{\text{disjoint balls}\}.$$



If there is not an outermost minimal surface, then M^3 is $\mathbb{R}^3$. Also, "outermost minimal surface" is well-defined by the maximum principle.



Exterior of Σ_1, Σ_2 forms $H<0$ barrier which implies the existence of another exterior minimal surface.

emma 3.8 (Hempel's "3-Manifolds")

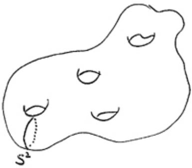
Cf $M^3 \setminus \{\text{embedded } S^2\}$ is connected, then

$$M^3 = N^3 \# (\text{an } S^2 \text{ bundle over } S^1).$$

xamples:

$S^2 \times S^1$:

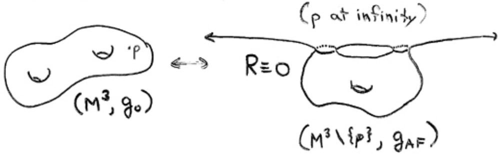
Some crazy manifold:



Proof:

Choose a path C in $M^3 \setminus S^2$ from one side of S^2 to the other. Let P^3 be the ε-neighborhood of $S^2 \cup C$. Note that ∂P^3 is a 2-sphere and that P^3 is an S^2 bundle over S^1 minus a ball. ∎

Given (M^3, g) let g_0 minimize E in $[g]$.
Given any $p \in M^3$, choose $g_{AF} \in [g]$ (on $M^3 \setminus \{p\}$)
such that



$(M^3 \setminus \{p\}, g_{AF})$ has zero scalar curvature, sends p to infinity, and is asymptotically flat. Note that if $g_{AF} = G(x)^4 \cdot g_0$, then $G(x)$ is the Green's function at p of the conformal Laplacian $\Delta_0 - \frac{1}{8} R_0$. Then

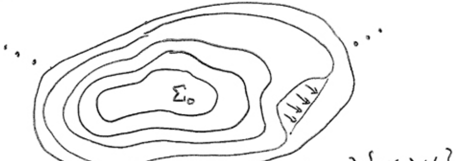
$$C(g) = C(g_0) = C(g_{AF}) = 8\, S(g_{AF})$$
$$\equiv \inf \left\{ \frac{\int_M 8 |\nabla u|^2}{\left(\int_M u^6\right)^{1/3}} \;\middle|\; u \in H^1(M, g_{AF}) \text{ and has compact support.} \right\}$$

Theorem 3: If $(M^3 \setminus \{p\}, g_{AF})$ has an outer-minimizing minimal S^2 which bounds a region, then

$$S(g_{AF}) \leq Y_2 / 8.$$

Theorem 2 follows from Theorem 3, which we now prove.

We use inverse mean curvature flow to prove Theorem 3. Given an outer-minimizing minimal S^2, Huisken - Ilmanen prove that we can use this as a starting point and define a weak IMCF for all $t \geq 0$.



$$\Sigma(t) = \partial\{\phi(x) < t\}$$

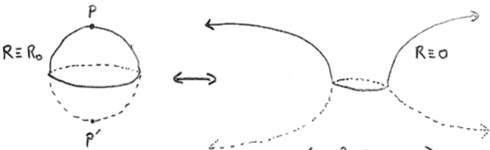
They define a real-valued function $\phi(x)$ such that its level sets $\Sigma(t)$ satisfy weak IMCF, meaning that for almost all $t \geq 0$, weak $H_{\Sigma(t)} = |\nabla \phi|_{\Sigma(t)}$.

Furthermore
$$|\Sigma(t)| = |\Sigma_0| \cdot e^t$$
and the Hawking mass
$$m_H(\Sigma(t)) = \sqrt{\frac{|\Sigma(t)|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma(t)} H^2\right)$$
is a nondecreasing function of t, even though $\Sigma(t)$ may occasionally jump.

Let
$$f(t) = \frac{1}{\sqrt{2e^{t} - e^{t/2}}}.$$

We claim that
$$u(x) \equiv f(\phi(x))$$
has Sobolev ratio
$$S(u) \equiv \frac{\int_M |\nabla u|^2}{\left(\int_M u^6\right)^{1/3}} \leq \frac{Y_2}{8} \quad \left(\text{in } (M^3 - \{p\}, g_{AF})\right).$$

This $f(t)$ comes from considering our construction on $(\mathbb{RP}^3, g_0)$ which we know has $C(g_0) = C(g_{AF}) = 8\,S(g_{AF}) = Y_2$.



$(\mathbb{RP}^3, g_0)$

(S^3, g_0) with antipodal points identified.

$(\mathbb{RP}^3 \setminus \{p\}, g_{AF})$

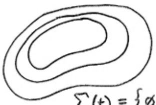
$(S^3 \setminus \{p, p'\}$, Schwarzschild metric) with antipodal points identified.

Ne estimate using the co-area formula.

$$dV = |\nabla \phi|^{-1} dA_{\Sigma(t)}\, dt$$

The co-area formula.

peed of
evel sets
$= |\nabla \phi|^{-1}$



$\Sigma(t) = \{\phi(x) = t\}$

lence, the numerator of the Sobolev ratio is

$$\begin{aligned}
\int_M |\nabla u|^2 dV &= \int_M \left| f'(\phi(x)) \cdot \nabla \phi \right|^2 dV \\
&= \int_0^\infty f'(t)^2 \int_{\Sigma(t)} |\nabla \phi|\, dA_{\Sigma(t)}\, dt \\
&= \int_0^\infty f'(t)^2 \left(\int_{\Sigma(t)} H\, dA \right) dt
\end{aligned}$$

(IMCF)

und the denominator is

$$\begin{aligned}
\int_M u^6 dV &\geq \int_0^\infty \int_{\Sigma(t)} f(\phi(x))^6\, |\nabla \phi|^{-1} dA\, dt \\
&= \int_0^\infty f(t)^6 \left(\int_{\Sigma(t)} H^{-1} dA \right) dt \\
&\geq \int_0^\infty f(t)^6\, |\Sigma(t)|^2 \left(\int_{\Sigma(t)} H\, dA \right)^{-1} dt \\
&= \int_0^\infty f(t)^6\, e^{2t}\, |\Sigma_0|^2 \left(\int_{\Sigma(t)} H\, dA \right)^{-1} dt.
\end{aligned}$$

(IMCF)

(Hölder's inequality)

By the monotonicity of the Hawking mass,

$$\sqrt{\frac{|\Sigma(t)|}{16\pi}}\left(1-\frac{1}{16\pi}\int_{\Sigma(t)}H^2\right)=m_H(\Sigma(t))\geq m_H(\Sigma_0)=\sqrt{\frac{|\Sigma_0|}{16\pi}},$$

so

$$\int_{\Sigma(t)}H^2 \leq 16\pi\left(1-\sqrt{\frac{|\Sigma_0|}{|\Sigma(t)|}}\right)$$

$$=16\pi\left(1-e^{-t/2}\right)$$

(Recall that $|\Sigma(t)|=|\Sigma_0|\cdot e^t$)

and

$$\int_{\Sigma(t)}H\,dA \leq |\Sigma(t)|^{1/2}\left(\int_{\Sigma(t)}H^2\,dA\right)^{1/2}$$

$$=\sqrt{16\pi\cdot|\Sigma_0|\cdot(e^t-e^{t/2})}$$

Thus, from our previous estimates we have

$$S(u)=\frac{\int_M|\nabla u|^2}{\left(\int_M u^6\right)^{1/3}}\leq\frac{(16\pi)^{2/3}\int_0^\infty f'(t)^2\sqrt{e^t-e^{t/2}}\,dt}{\left(\int_0^\infty f(t)^6 e^{2t}(e^t-e^{t/2})^{-1/2}\,dt\right)^{1/3}}$$

(Recall that $f(t)=\frac{1}{\sqrt{2e^t-e^{t/2}}}$)

$$=\ ?/8\ !$$

We don't actually have to compute the above integrals. We know that we have equality in all of our inequalities for $(\mathbb{RP}^3\setminus\{p\}$, Schwarzschild) and $S(u)=\ ?/8$ in this case. Thus, Theorem 3 follows.

Open Problems
(Homework)

. If g_0 is a constant (positive, negative) curvature metric on M^3, then

$$Y(M^3) = E(g_0).$$

he negative case follows from the following ery beautiful conjecture.

2. Suppose M^3 is closed and admits a hyperbolic metric g_0. Then for any other metric g on M^3,

$$R(g) \geq R(g_0) \Rightarrow Vol(g) \geq Vol(g_0).$$

his volume comparison conjecture, known as the iromov/Schoen conjecture, is known for Ricci curvature (Besson-Courtois-Gallot). Also, it is alse in the corresponding S^3/positive case.

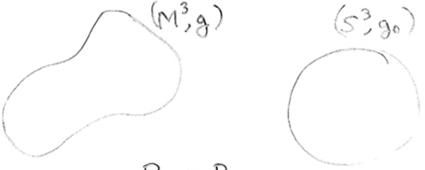
3. What is $Y(\mathbb{RP}^3 \# (S^2 \times S^1))$?

We know it is in $[Y_2, Y_1]$.

4. Prove that all other prime 3-manifolds other than $\underbrace{S^3, S^2\times S^1, S^2 \tilde{\times} S^1}_{Y_1}, \underbrace{\mathbb{RP}^3, \mathbb{RP}^2\times S^1}_{Y_2}$ have $Y < Y_2$.

5. $Y > 0$ implies M^3 is a connect sum of quotients of S^3 and quotients of $S^2 \times S^1$.

Theorem (thesis)



$$R_g \geq R_{g_0}$$

$$Ric_g \geq \varepsilon_0 \, Ric_{g_0}$$

$$\Downarrow \qquad (\varepsilon_0 = .134\ldots)$$

$$Vol(g) \leq Vol(g_0)$$