ecture 1 (Tuesday, August 5, 2002)

"Generalization of the Hawking mass"

ecture 2 (Wednesday)

"Proof of the Poincare Conjecture for 3-Manifolds with Yamabe Invariant Greater Than $\mathbb{RP}^3$"

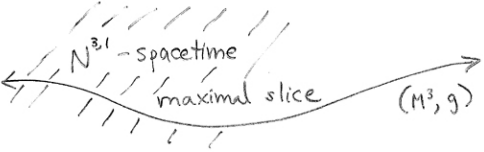
ecture 3 (Friday)

"Black Holes, the Penrose Conjecture, and Quasi-Local Mass"

Hubert Bray, MIT

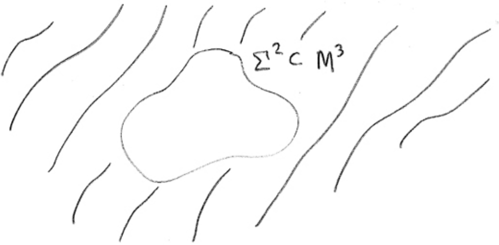
"50 Years of the Cauchy Problem in General Relativity" July 29 - August 10, 2002
Institut D'Études Scientifiques De Cargèse
Cargèse, Corsica

"Generalization of the Hawking Mass"



Dominant Energy Condition $\Rightarrow$ $R^M \geq 0$.

<u>Problem</u>: How do we explicitly estimate how much mass is inside a surface Σ^2 in M^3?



Based on the spherically symmetric case, Hawking wrote down the estimate

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\right)$$

where H is the mean curvature of Σ^2 in M^3 and $|\Sigma|$ is the area of Σ. Given sufficient asymptotics, this expression converges to the total ADM mass on large, round spheres going to infinity.

If $\Sigma^2 \subset (\mathbb{R}^3, \delta)$, note that

$$H^2 = (\lambda_1 + \lambda_2)^2 = 4\lambda_1\lambda_2 + (\lambda_1 - \lambda_2)^2$$
$$= 4K + (\lambda_1 - \lambda_2)^2$$

where λ_1, λ_2 are the principle curvatures of Σ^2. By Gauss-Bonnet,

$$\int_\Sigma H^2 = \int_\Sigma 4K + (\lambda_1 - \lambda_2)^2$$
$$= 16\pi + \int_\Sigma (\lambda_1 - \lambda_2)^2 \geq 16\pi,$$

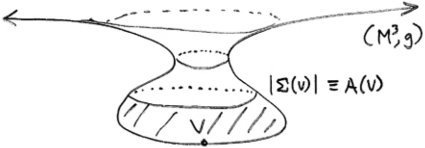
n spheres, so

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2\right) \leq 0.$$

So the Hawking mass always gives a lower bound for $m_{ADM} = 0$ in $(\mathbb{R}^3, \delta)$.

Kind of cool .... but hardly convincing of any greater relevance.

Also, in the spherically symmetric case,



(M^3, g) is characterized by $A(v)$. Furthermore,

$$A'(v) = \frac{A'(t)}{V'(t)} = \frac{\int_{\Sigma(v)} H}{\int_{\Sigma(v)} 1} = H(v) \qquad \text{(using unit speed flow)}$$

so

$$m_H(\Sigma(v)) = \sqrt{\frac{|\Sigma(v)|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma(v)} H^2\right)$$
$$= \sqrt{\frac{A(v)}{16\pi}}\left(1 - \frac{1}{16\pi}\cdot A(v) A'(v)^2\right)$$

and

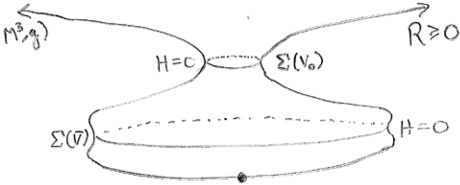
$$\frac{d}{dv} m_H(\Sigma(v)) = \left(\frac{A(v)}{16\pi}\right)^{1/2}\left(\frac{A'(v)}{16\pi}\right)\cdot R(v)$$

where the scalar curvature

$$R(v) = \frac{8\pi}{A} - \frac{3}{2}A'(v)^2 - 2A\cdot A''(v).$$

Hence, in the spherically symmetric case we can prove the Positive Mass Theorem and the Penrose Inequality (in this setting).

$$m_{ADM} = \lim_{V \to \infty} m_H(\Sigma(V)) \geq m_H(\Sigma(V_0)) = \sqrt{\frac{|\Sigma(V_0)|}{16\pi}}$$



Since $\frac{d}{dV} m_H(\Sigma(V)) = \left(\frac{A}{16\pi}\right)^{1/2} \left(\frac{A'(V)}{16\pi}\right) R(V) \geq 0$

Note that the horizon $\Sigma(V_0)$ must be outerminizing for the Penrose Inequality.

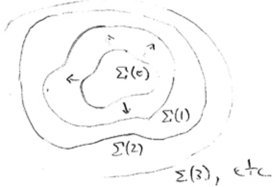
So maybe this might work in the non-spherical case!

Amazing Miracle Happens! (mid seventies)

Geroch - Jang - Wald observe that if you flow Σ^2 in the outward normal direction with speed $\eta = \frac{1}{H}$, then

$$\frac{d}{dt} m_H(\Sigma(t)) \geq 0 .$$

Picture:

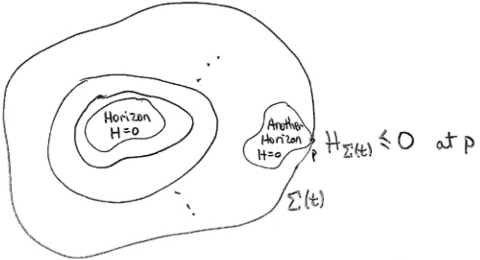


$\Sigma(t)$, $t \geq 0$ is family of surfaces generated.

Hence,

$$\text{"}\; m_{ADM} = \lim_{t\to\infty} m_H(\Sigma(t)) \geq m_H(\Sigma(0)). \;\text{"}$$

Great! But there is a problem. How do you flow $\Sigma(t)$ with speed $\eta = \frac{1}{H}$ if $H=0$? What if H changes sign on $\Sigma(t)$? For example, consider



and $H_{\Sigma(t)} < 0$ must occur somewhere on $\Sigma(t)$ by the maximum principle.

Oh well, its a nice heuristic argument anyway ...

Another Amazing Miracle Happens! (1997)

Huisken - Ilmanen show that this argument can actually be made rigorous.

Geometric Trick: Whenever $\Sigma(t)$ is enclosed by a surface of equal or less area, "jump" to that surface and continue flowing with $\eta = \frac{1}{H}$. This rule guarantees "$H>0$" so that η is defined. Also note that during a jump,

- Area does <u>not</u> decrease (because otherwise it would have jumped earlier)
- $\int_\Sigma H^2$ decreases, so
- $m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H^2\right)$ does not decrease.

Question: Are there other explicit quasi-local mass functionals which give lower bounds on the ADM mass?

Answer: Yes, perhaps many. However, none were found until March 2002. The most explicit example is

$$m_{ADM} \geq m_H(\Sigma) + \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \cdot \frac{\max\left(0, \int_\Sigma H^3 - \frac{(16\pi)^{3/2}}{|\Sigma|^{1/2}}\right)^2}{6 \int_\Sigma H^4}$$

where as usual

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_\Sigma H^2\right).$$

But first, let's look for examples of the form

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma f(H)\right)$$

We already know that $f(x) = x^2$ works since that gives the Hawking mass. More generally, we observe that if we consider the flow

$$\eta = \frac{2}{f'(H)},$$

then Gauss-Bonnet, 1st and 2nd variation formulas for area, Gauss equation, yaddy yaddy yadda imply

$$\frac{d}{dt}\int_\Sigma f(H) \leq 8\pi + \int_\Sigma \frac{2Hf(H)}{f'(H)} - \frac{3}{2}H^2.$$

If we choose $y=f(x)$ to satisfy

$$(*) \quad \frac{dy}{dx} = \frac{4xy}{3x^2-y}, \qquad (f(0)=0)$$

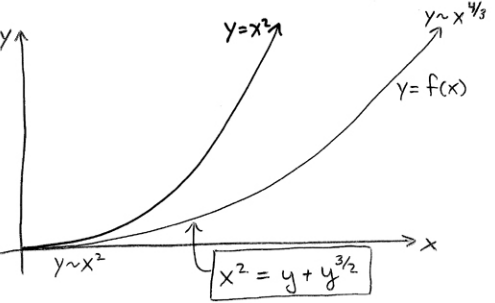
then
$$\frac{d}{dt}\int_\Sigma f(H) \leq 8\pi - \frac{1}{2}\int_\Sigma f(H),$$

which, together with
$$\frac{d}{dt}|\Sigma| \geq |\Sigma|$$
(which we are not checking) yields

$$\frac{d}{dt} m(\Sigma) = \frac{d}{dt}\sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma f(H)\right) \geq 0$$

under $\eta = \dfrac{2}{f'(H)}$ flow (when $m(\Sigma)\geq 0$).

Clearly $f(x)=x^2$ satisfies $(*)$. Are there any other solutions?



$y=f(x)$ solves $\frac{dy}{dx} = \frac{4xy}{3x^2-y}$ as does $y=\frac{1}{c^2}f(cx)$ for any $c>0$. Note that $c\to 0$ gives $y=x^2$. Thus,

$$m_c(\Sigma) \equiv \sqrt{\frac{|\Sigma|}{16\pi}}\left(1-\frac{1}{16\pi}\int_\Sigma \frac{1}{c^2} f(cH)\right)$$

is monotone for this choice of f and any $c>0$, under $\eta = \frac{2c}{f'(cH)}$ flow.

But does $\lim m_c(\Sigma(t)) = m_{ADM}$?

No! But this mass functional does converge to $m_{ADM} + \underline{\underline{c}}$. Hence, we redefine

$$m_c(\Sigma) \equiv \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \frac{1}{c^2} f(cH)\right) - c$$

so that $m_c(\Sigma(t))$ is now monotone under

$$\eta = \frac{2c}{f'(cH)}$$

flow and

$$\lim_{t\to\infty} m_c(\Sigma(t)) = m_{ADM}.$$

Thus,

$$m_{ADM} \geq m_c(\Sigma), \quad \forall c > 0.$$

Note that $c = 0 \Leftrightarrow$ Hawking mass and inverse mean curvature flow.

Define

$$\tilde{m}(\Sigma) = \sup_{c>0} m_c(\Sigma)$$

Then we have

$$m_{ADM} \geq \tilde{m}(\Sigma) \geq m_H(\Sigma)$$

since $c=0 \Leftrightarrow$ Hawking mass. Thus we have a better lower bound for the ADM total mass. Estimating

$$f(x) \leq x^2 - x^3 + \frac{3}{2}x^4$$

gives

$$\tilde{m}(\Sigma) \geq \sup_c \sqrt{\frac{|\Sigma|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma \left(H^2 - cH^3 + \frac{3}{2}c^2H^4\right)\right) - c$$

$$= m_H(\Sigma) + \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \cdot \frac{\max\left(0, \int_\Sigma H^3 - \frac{(16\pi)^{3/2}}{|\Sigma|^{1/2}}\right)^2}{6\int_\Sigma H^4}$$

from before.

Next time we will use inverse mean curvature flow to prove a topological result about the Yamabe invariant Y.

Thm. A closed 3-manifold with $Y > Y(\mathbb{RP}^3)$ is either S^3 or a connect sum with an S^2 bundle over S^1.

We will also compute $Y(\mathbb{RP}^3)$, $Y(\mathbb{RP}^2 \times S^1)$ and show that the first five prime 3-manifolds ordered by Y are

S^3, $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $\mathbb{RP}^3$, and $\mathbb{RP}^2 \times S^1$.