Holonomy of Lorentzian manifolds

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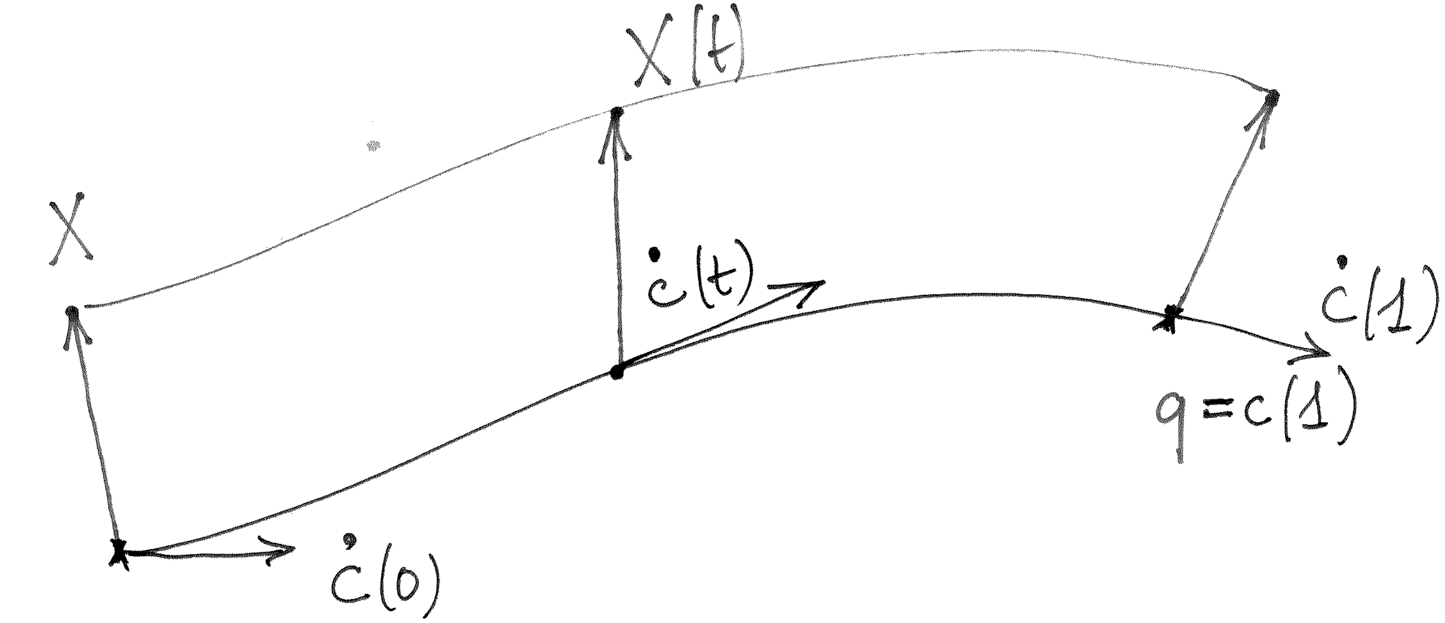
• Let ∇ be a linear connection on a manifold M.



$p = c(0)$ $\qquad\qquad$ $\nabla_{\dot c(t)} X(t) = 0$

$$\tau_c : T_pM \longrightarrow T_qM .$$

• If c is a <u>loop</u> $(c(0) = p = q = c(1))$, $\tau_c : T_pM \longrightarrow T_pM$

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- $\langle \tau_c \mid c \text{ loop at } p \rangle = Hol_p(M, \nabla) \subset Gl(T_pM)$

É. CARTAN 1926 holonomy representation Lie group

A. BOREL – A. LICHNEROWICZ 1952

- In the context of bundle theory, one realizes that to occur a holonomy representation is constrained by topological restrictions.

$\longrightarrow$ One works with the restricted holonomy.

- Quantities invariant by the holonomy

→ Let g be a non degenerate metric.

If ∇ is the Levi-Civita connection, $\nabla g = 0$.

→ τ_c preserves $g \Rightarrow \mathrm{Hol}_p(\nabla) \subset O(T_pM, g_p)$

→ Any parallel geometric structure reduces the holonomy.

- A. AMBROSE – I.M. SINGER 1953

$$\mathrm{Hol}_p(\nabla) = \langle R_{X,Y} \mid X, Y \in T_pM \rangle$$

$$\tau_c^{-1} \circ \qquad \circ\, \tau_c \qquad q$$

$\longrightarrow$ Torsion(∇) brings a constraint to holonomy.

M. BERGER 1955 1^{st} condition

in the torsion free case

Let $T_pM = V$.

Let $\mathfrak{h} \subset \mathfrak{gl}(V)$. Then, if $\mathrm{Hol}_p(\nabla) = \mathfrak{h}$,

$K(\mathfrak{h}) \neq K(\mathfrak{g})$ for any proper ideal $\mathfrak{g} \subset \mathfrak{h}$

$\left(K(\mathfrak{h}) = \text{kernel} : \delta : \mathfrak{h} \otimes \Lambda^2 V^* \longrightarrow V \otimes \Lambda^3 V^* \right)$

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$\longrightarrow$ Connection with | the Ricci curvature
| the Einstein equation

- Irreducibility and indecomposability

(M, g) is <u>irreducible</u> if $Hol_p(M)$ is irreducible.

(M, g) is <u>indecomposable</u> if $Hol_p(M)$ does not leave any proper non trivial subspace of $T_p M$ invariant.

A. BOREL - A. LICHNEROWICZ 1952
local decomposability theorem

G. DE RHAM 1952 global version H. WU 1964

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THEOREM Let (M, g) be a simply connected complete pseudo-Riemannian manifold. Then (M, g) is isometric to a product of simply connected complete indecomposable pseudo-Riemannian manifolds. Indeed, if $T_pM = E_0 \oplus E_1 \oplus \dots \oplus E_r$ is an orthonormal decomposition of T_pM in non trivial subspaces such that the holonomy representation is trivial in E_0 and indecomposable in each E_i, then there exist r simply connected pseudo-Riemannian manifolds $(N_i, g_i)_{1 \leq i \leq r}$ so that

$$(M, g) \underset{\text{isom.}}{\simeq} (\mathbb{R}^a, \pm dt^2) \times (N_1, g_1) \times \dots \times (N_r, g_r)$$

- Classification in the Riemannian case

M. BERGER 1955

<u>THEOREM</u> 1) Let (M, g) be an irreducible Riemannian manifold which is <u>not locally symmetric</u>. Then the holonomy is one of the following

SO_n

SU_m or U_m provided $n = 2m$

Sp_q or $Sp_1 Sp_q$ provided $n = 4q$

G_2 provided $n = 7$

$Spin_7$ provided $n = 8$

2) For the Lorentzian case, one is left with $SO^0_{n-1,1}$ for the irreducible non symmetric case.

- The symmetric case: (always <u>irreducible</u>)

→ Riemannian

J. SIMONS 1962

$Hol_p(g)$ acts transitively on the sphere $\Rightarrow$ semi-simplicity

→ Lorentzian

M. CAHEN – N.R. WALLACH 1970

symmetric [redacted]

<u>THEOREM</u> Indecomposable/irreducible Lorentzian manifolds are locally isometric to one of the following spaces M_φ

$$M_\varphi = (G/K, g) \qquad \varphi : \mathbb{R}^n \to \mathbb{R}^n \text{ symmetric}$$

G solvable $\quad \mathfrak{g} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \quad K = \mathbb{R}^n$

g induced from $0 \oplus e_2 \oplus h$

• Understanding the case of <u>invariant null lines</u> is the key for the Lorentzian setting.

L. BERARD-BERGERY - A. IKEMAKHEN 1993

A parallel line leads either to a parallel vector fiel or to a recurrent vector field.

$$\nabla_Y X = f(Y)X \; .$$

• Classification for 4-dimensional Space-times

G.S. HALL - D.P. LONIE 2000

Type	Lie algebra	Recurrent v.f.	Parallel v.f.	Dim.
R_2	$l \wedge n$	l, n	$\langle x, y \rangle$	1
R_3	$l \wedge x$	—	$\langle l, y \rangle$	1
R_4	$x \wedge y$	—	$\langle l, n \rangle$	1
R_6	$l \wedge n, l \wedge x$	l	$\langle y \rangle$	2
R_7	$l \wedge n, x \wedge y$	l, n	—	2
R_8	$l \wedge x, l \wedge y$	—	$\langle l \rangle$	2
R_9	$l \wedge n, l \wedge x, l \wedge y$	l	—	3
R_{10}	$l \wedge n, l \wedge x, n \wedge x$	—	$\langle y \rangle$	3
R_{11}	$l \wedge x, l \wedge y, x \wedge y$	—	$\langle l \rangle$	3
R_{12}	$l \wedge x, l \wedge y, l \wedge n + e(x \wedge y)$	l	—	3
R_{13}	$x \wedge y, y \wedge z, x \wedge z$	—	$\langle u \rangle$	3
R_{14}	$l \wedge n, l \wedge x, l \wedge y, x \wedge y$	l	—	4
R_{15}		—	—	6

THEOREM If M^4 is simply connected, and g, non flat,

→ satisfies the vacuum equations, $Hol_p(M,g)$ is either R_8, R_{14} or R_{15};

→ Satisfies the Einstein equations, $Hol_p(M,g)$ is either R_7, R_{14} or R_{15};

→ Satisfies the conformally flat condition, $Hol_p(M,g)$ is either R_7, R_8, R_{10}, R_{13}, R_{14} or R_{15};

→ and the energy-momentum tensor is of null Einstein-Maxwell type, $Hol_p(M,g)$ is either R_3, R_8, R_{10}, R_{14} or R_{15};

→ and the energy-momentum tensor is of non-null Einstein-Maxwell type, $Hol_p(M,g)$ is either R_7, R_{14} or R_{15};

→ and the energy-momentum tensor represents a perfect fluid, $Hol_p(M,g)$ is either R_{10}, R_{13} or R_{15};

→ and the energy-momentum tensor represents massive scalar fields, $Hol_p(M,g)$ is R_{15}.

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- Some open questions

→ Classification in any dimension

→ Refinements that may be worth exploring:
* definition of invariant objects in terms of spinors.
* Spatial holonomy?
* null holonomy?

→ Twistor techniques of S. MERKULOV - L. SCHWACHHÖFER