Cheeger-Gromov Theory and Applications to General Relativity

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§4. Applications to Static and Stationary Space-Times.

Apply convergence/collapse theory to (vacuum) stationary space-times $(\mathbf{M},\ \mathbf{g})$.

Time-independent - so can use directly methods of Riemannian geometry.

K = time-like Killing field. Assume throughout

$$(\mathbf{M},\ \mathbf{g}) \text{ chronological, } K \text{ complete.}$$

Σ = orbit space of $\mathbb{R}$-action, $\pi : \mathbf{M} \to \Sigma$ principal $\mathbb{R}$ bundle.
4-metric $\mathbf{g}$ of form:

$$\mathbf{g} = -u^2(dt + \theta)^2 + \pi^*(g); \tag{4.1}$$

$K = \partial/\partial t$, θ connection 1-form for bundle π, $u^2 = -\mathbf{g}(K, K) > 0$ and $g = g_\Sigma$ = metric induced on orbit space.

Define twist 1-form ω by

$$2\omega = *(\kappa \wedge d\kappa) = -u^4 * d\theta,$$

$\kappa = -u^2(dt + \theta)$, 1-form dual to K.

The vacuum Einstein equations

$$Ric_{\mathbf{g}} = 0,$$

equivalent to elliptic system of P.D.E's in the data (g, u, ω):

$$Ric_g = u^{-1}D^2u + 2u^{-4}(\omega \otimes \omega - |\omega|^2 g), \tag{4.2}$$

$$\Delta u = -2u^{-3}|\omega|^2, \tag{4.3}$$

$$d\omega = 0. \tag{4.4}$$

Σ open, possibly with boundary. Locally, to obtain uniqueness, need to impose boundary conditions.

• Global issues.

(Σ, g) = complete, non-compact 3-manifold.

Boundary conditions then at infinity.

Theorem 4.1 (Lichnerowicz). *The only complete, stationary vacuum space-time (* $\mathbf{M}, \mathbf{g}$ *) which is asymptotically flat (AF) is empty Minkowski space-time* $(\mathbf{R}^4, \eta)$.

Stationary space-times model isolated physical systems. Only physically realistic models are AF.

• Physically, Lichnerowicz theorem a triviality. No source for the gravitational field, it must be empty.

• Mathematically, not (so) trivial. In fact

AF assumption ad-hoc, counter to spririt of GR

Theorem 4.2 (Generalized Lichnerowicz). *The only complete stationary vacuum space-time (* $\mathbf{M}, \mathbf{g}$ *) is empty Minkowski space-time* $(\mathbf{R}^4, \eta)$*, or a discrete isometric quotient of it.*

Outline of Proof:

• Study moduli space of complete stationary vacuum solutions, any asymptotic behavior.

Curvature could be unbounded at infinity. If so, can find base points and rescalings to obtain new stationary vacuum solution, (i.e. a new point in the moduli space), with

$$|Ric_{\Sigma}|(x) = 1, \ |Ric_{\Sigma}|(y) \leq K, \forall y.$$

• Ernst formulation. Define the conformally related metric $\widetilde{g}$:

$$\widetilde{g} = u^2 g. \tag{4.5}$$

Equation for Ric then becomes:

$$Ric_{\widetilde{g}} = 2(d \ln u)^2 + 2u^{-4}\omega^2 \geq 0. \tag{4.6}$$

Elliptic system (4.2)-(4.4) becomes the Euler-Lagrange equations for

$$S_{\text{eff}} = \int [R - \frac{1}{2}(\frac{|d\phi|^2 + |du^2|^2}{u^4})],$$

$\phi =$ twist potential, $d\phi = 2\omega$.

3-d gravity coupled to σ-model, target $= (H^2(-1), g_{-1})$.

Thus, Ernst map

$$E = (\phi, u^2) \tag{4.7}$$

is harmonic map

$$E : (\Sigma, \widetilde{g}) \to (H^2(-1), g_{-1}).$$

Harmonic maps $E : (M, g) \to (N, h)$ with domain $Ric \geq 0$, target of $curv \leq 0$ have strong rigidity properties, via the Bochner-Lichnerowicz formula,

$$\frac{1}{2}\Delta|DE|^2 = |D^2E|^2 + < Ric_g, E^*(h) > - \sum(E^*R_h)(e_i, e_j, e_j, e_i). \tag{4.8}$$

Analyse this carefully:

show E is a constant map, and so $(\mathbf{M},\ \mathbf{g})$ is flat.

Remark 4.3

(i). Same result and proof holds for stationary gravitational fields coupled to σ-models, with target spaces = Riemannian manifolds of non-positive sectional curvature.

(ii). Open Problem.

Riemannian analogue of generalized Lichnerowicz.

Thus, does there exist a non-flat Ricci-flat Riemannian 4-manifold which admits a *free* isometric S^1 action?

• Local issues.

This rigidity $\Rightarrow$ apriori estimates on geometry of general stationary (vacuum) solutions.

Suppose Σ not complete, so $\partial\Sigma \neq \emptyset$.

Part of $\partial\Sigma$ may correspond to horizon $H = \{u = 0\}$.

Theorem 4.4 (Curvature Estimate). *Let* $(\mathbf{M}, \mathbf{g})$ *be a stationary vacuum space-time. Then there is a constant* $C < \infty$, *independent of* $(\mathbf{M}, \mathbf{g})$, *such that*

$$|\mathbf{R}|(x) \leq C/r^2[x], \tag{4.9}$$

where $r[x] = dist_\Sigma(\pi(x), \partial\Sigma)$.

Here, the curvature norm $|\mathbf{R}|$ may be given by

$$|\mathbf{R}| = |R_\Sigma| + |d\ln u|^2 + |u^{-2}\omega|^2.$$

Remark 4.5 (i). Using elliptic regularity, one also has higher order bounds:

$$|\nabla^k\mathbf{R}|(x) \leq C_k/r^{2+k}[x]. \tag{4.10}$$

(ii). A version of this result also holds for stationary space-times with energy-momentum tensor T. Thus, for example one has

$$|\mathbf{R}|(x) \leq C_\alpha \cdot |T|_{C^\alpha(B_{[x]}(1))}, \tag{4.11}$$

for any $\alpha > 0$, where $B_{[x]}(1)$ is the unit ball in (Σ, g) about $[x]$.

Thus, one can use the Cheeger-Gromov theory to control local behavior of stationary space-times, away from any boundary.

• Asymptotic behavior.

Study apriori possible asymptotic behavior of stationary vacuum solution. Know for instance,

curvature decays as r^{-2} at ∞.

Restrict to static space-times $(\mathbf{M},\ \mathbf{g})$

Orbit space (Σ, g). Define $\partial\Sigma$ to be *pseudo-compact* if $\exists r_o > 0$ s.t. level set $\{r = r_o\}$ in Σ is compact.

Let $S(s) = r^{-1}(s) \subset \Sigma$. If E is an end of (Σ, g), set

$$m_E = lim_{s\to\infty}\tfrac{1}{4\pi}\int_{S(s)} < \nabla \ln u, \nabla t > dA. \tag{4.12}$$

Theorem 4.6 (Static Asymptotics). *$(\mathbf{M},\ \mathbf{g})$ a static vacuum space-time with pseudo-compact boundary. Then*
- *$(\mathbf{M},\ \mathbf{g})$ has a finite number of ends.*
- *Any end E on which $\liminf_E u > 0$, is either:*

AF

or

$$\text{small} \equiv \int_1^\infty areaS(r)^{-1}dr = \infty. \tag{4.13}$$

- *If $m_E \neq 0$ and $sup_E u < \infty$, then E is AF.*

When E is AF, it is AF in usual "strong" sense:

$$|g - g_0| = \frac{2m}{r} + O(r^{-2}), \; |R| = O(r^{-3}), \quad |u - 1| = \frac{m}{r} + O(r^{-2}).$$

Ideas of Proof:

Study asymptotic behavior of an end E by "blowing it down".

R large, k fixed: consider annuli $A(R, kR)$ about $x_o \in (\Sigma, g)$. In rescalings

$$g_R = R^{-2} g,$$

$A(R, kR)$ becomes metric annulus $A(1, k)$ w.r.t. g_R. Quadratic curvature decay $\Rightarrow$ curvature of g_R uniformly bounded.

Thus, apply the Cheeger-Gromov theory to a sequence $(A(1, k), g_{R_i})$, $R_i \to \infty$.

Convergence (or cusp) case gives AF ends, collapse case gives small ends.

For small ends, obtain an extra S^1 or T^2 symmetry when collapse is unwrapped in covering spaces. Asymptotic behavior then described by axisymmetric static solutions, i.e. the Weyl metrics.

Remark 4.7 There exist static vacuum solutions, *smooth* up to the horizon, which have a single *small* end.

Myers metrics = periodic Schwarzschild metrics.

$\Sigma = (D^2 \times S^1) \setminus S^2$, $\partial\Sigma = S^2$, end $T^2 \times \mathbb{R}^+$, asymptotic to a (static) Kasner metric.

This is of course not a counterexample to the static black hole uniqueness theorem, since the end is not AF.

§5. Lorentzian Analogues and Open Problems

Issue: Apply convergence/collapse theory to time-independent space-times $(\mathbf{M}, \mathbf{g})$.
Main Focus: Vacuum space-times

$$Ric_{\mathbf{g}} = 0,$$

or at least $|Ric_{\mathbf{g}}| \leq K$.

• Motivation Global stability results:

1. Minkowski space-time (Christodoulou-Klainerman)
2. de Sitter space-time (Friedrich)
3. Milne space-time (Andersson-Moncrief)
4. $U(1)$ Bianchi model (Choquet-Bruhat & Moncrief)

openness results

Basic features of given model preserved under (suitable) small perturbations of initial data.
What happens to limits of such perturbations?

• Difficulties.
(i). Ricci curvature:

Elliptic PDE for Riem. metrics $\rightarrow$ Hyperbolic PDE for Lor. metrics

(ii).

$$O(4) \text{ compact} \rightarrow O(3,1) \text{ non-compact}$$

1^{st} Level Problem

Bound local geometry in terms of

$$|\mathbf{R}|_{L^\infty} \leq K. \tag{5.1}$$

Usual norm of curvature tensor

$$|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \ \textsf{not} \ \geq 0. \tag{5.2}$$

For Ricci-flat Lorentz metrics, 2 scalar invariants of full curvature

$$|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \text{ and } < \mathbf{R}, *\mathbf{R} >= \mathbf{R}_{ijkl}(*\mathbf{R}^{ijkl}).$$

Both can vanish on non-flat space-times: e.g.

plane-fronted gravitational waves

$$\mathbf{g} = -2dudv + 2(dx^2 + dy^2) - 2h(u,x,y)du^2,$$

$$\Delta_{(x,y)}h = 0, \ h \text{ arbitrary in } u.$$

Class of such highly non-compact ⇒ no local control of metric in any coord. system, under bounds (5.2).

Thus, must consider bounds on $\mathbf{R}$ w.r.t. a framing or coordinate system.

Let $T = e_0 =$ unit time-like vector, future directed. Extend to o.n. frame e_α, $0 \leq \alpha \leq 3$. $T^\perp$ space-like, $O(3)$ compact, so framing of $T^\perp$ unimportant. Define

$$|\mathbf{R}|_T^2 = \sum (\mathbf{R}_{ijkl})^2; \tag{5.3}$$

components w.r.t. frame e_α. Equivalent to taking the norm of $\mathbf{R}$ w.r.t. the Riemannian metric

$$g_E = \mathbf{g} + 2T \otimes T.$$

As long as T stays in a compact subset of $T^+\mathbf{M} =$ bundle of future interior null cones, norms (5.3) all equivalent.

Noted earlier:
$(M, g) =$ smooth Riemannian manifold with L^∞ bound on the full curvature,

$$|R| \leq K$$

then $\exists$ charts in which have $L^{2,p}$ control of metric; bounds on $g_{\alpha\beta}$ and radius of charts depend only on K and lower volume bound.

Size conditions. Let Ω = domain in a smooth Lorentz manifold $(\mathbf{M},\ \mathbf{g})$, with smooth time function $T = \partial/\partial t$. Let $S = t^{-1}(0)$ and suppose the 1-cylinder

$$C_1 = B_p(1) \times [-1, 1] \subset\subset \Omega.$$

Let $D = ImT|_{C_1} \subset\subset T^+\Omega$.

Theorem 5.1 *Suppose Ω satisfies the size conditions and $\exists$ constants $K < \infty$, $v_o > 0$ s.t.*

$$|\mathbf{R}|_T \leq K, \quad vol_g B_p(1) \geq v_o. \tag{5.4}$$

Then $\exists r_o > 0$, $R_o < \infty$, depending only on K, v_o and D, and coord. charts on the r_o-cylinder

$$C_{r_o} = B_p(r_o) \times [r_o, r_o] \subset C_1,$$

s.t. on C_{r_o},

$$||\mathbf{g}_{\alpha\beta}||_{L^{2,p}} \leq R_o. \tag{5.5}$$

Result formulated so easy to pass to limits:

Given sequence of smooth space-times $(\mathbf{M}_i, \mathbf{g}_i)$ satisfying hypotheses of Theorem.

Then, in a subsequence, $\exists$ limit $C^{1,\alpha} \cap L^{2,p}$ space-time $(\mathbf{M}_\infty, \mathbf{g}_\infty)$, defined at least on the cylinder C_{r_o}.

Further, the convergence to the limit is $C^{1,\alpha}$ and weak $L^{2,p}$.

Result holds in dim $= n + 1$.

Idea of Proof: Construct a new time function τ on a small cylinder C_{r_o} with $|\nabla\tau|^2 = -1$, so the flow of $\nabla\tau$ is by time-like geodesics. On the level sets S_τ of τ, construct local harmonic coordinates x_1, x_2, x_3, (w.r.t. induced Riemannian metric).

Gives local coordinate system (τ, x_1, x_2, x_3) on C_{r_o}. Use Raychaudhuri equation, Bochner-Weitzenbock formula, (Simons' equation), and elliptic estimates to control $\mathbf{g}_{\alpha\beta}$.

If lower volume bound on $B_p(1) \subset S$ is dropped, then S may collapse with bounded curvature. Examples of this behavior occur on approach to Cauchy horizons, (Taub-NUT example).

Rendall: if Σ is a *compact* Cauchy surface in say smooth vacuum space-time, then nearby space-like hypersurfaces collapse with bounded curvature.