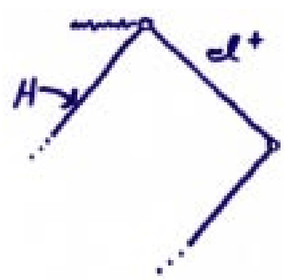
C^0 Null hypersurfaces

In GR, the null hypersurfaces of interest, e.g. horizons of various sorts, are not smooth in general.

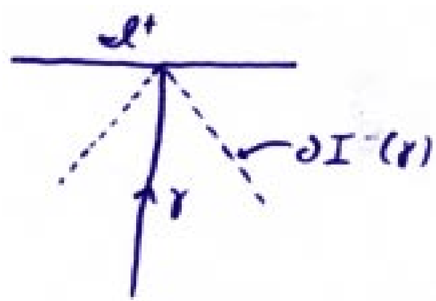
Such hypersurfaces often arise as the null portions of achronal boundaries, i.e., boundaries of pasts/futures,

$$A \subset M, \qquad \partial I^{\pm}(A) = \text{ achronal boundary.}$$

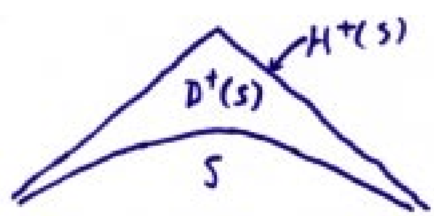
- Black hole event horizon: $H = \partial I^{-}(\mathcal{J}^{+}) \cap M$



- Observer horizons: $\partial I^{-}(\gamma)$

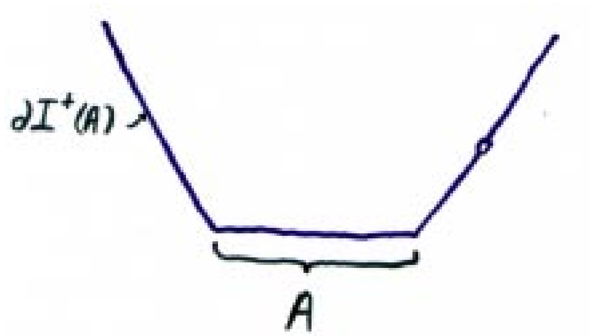


- Cauchy horizons: $H^{+}(S) = \partial I^{-}(D^{+}(S)) \cap J^{+}(S)$



Achronal boundaries.

Def. An achronal boundary is a set of the form $\partial I^+(A)$ (or $\partial I^-(A)$).



Prop. An achronal boundary $\partial I^+(A)$, if nonempty, is a closed achronal C^0 hypersurface in M.

Discussion of proof:

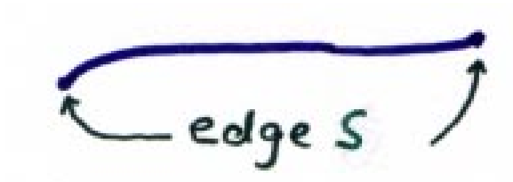
Lemma. *If $p \in \partial I^+(A)$ then $I^+(p) \subset I^+(A)$, and similarly, $I^-(p) \subset M \setminus \overline{I^+(A)}$.*

Pf: $q \in I^+(p) \Rightarrow p \in I^-(q)$. Since $I^-(q)$ is a nbd of p, and p is on the boundary of $I^+(A)$, $I^-(q) \cap I^+(A) \neq \emptyset$, and hence $q \in I^+(A)$.

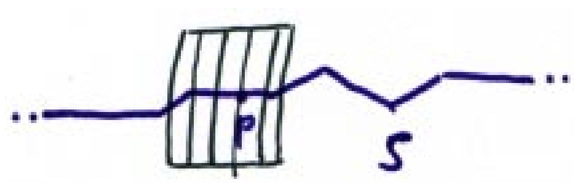
Since $I^+(A)$ does not meet $\partial I^+(A)$, it follows from the lemma that $\partial I^+(A)$ is achronal.

It also follows from the lemma that $\partial I^+(A)$ is edgeless.

Def. The edge of a closed achronal set $S \subset M$ is the set of points $p \in S$ such that every neighborhood U of p, contains a timelike curve from $I^-(p,U)$ to $I^+(p,U)$ that does *not* meet S.

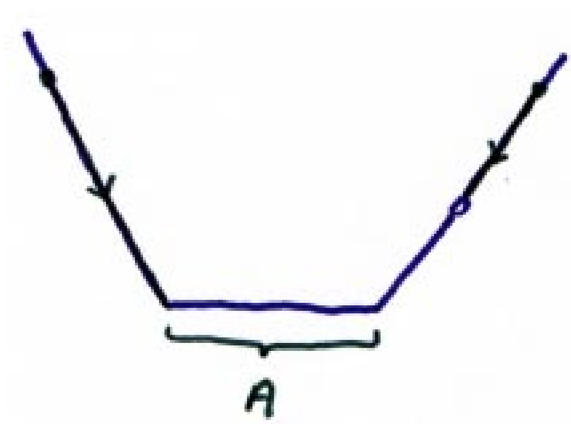


Prop. *A closed achronal edgeless set $S \subset M$ is a C^0 hypersurface in M.*

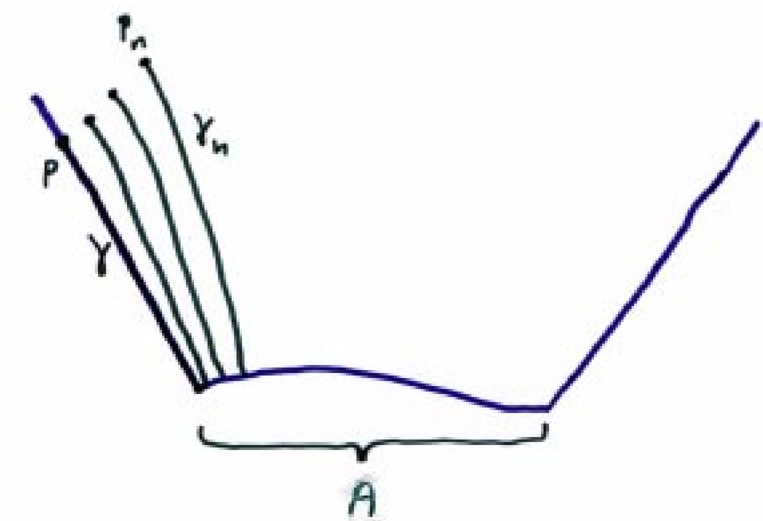


As a corollary, achronal boundaries are C^0 hypersurfaces.

Prop. *Let $A \subset M$ be closed. Then each $p \in \partial I^+(A) \setminus A$ lies on a null geodesic contained in $\partial I^+(A)$, which either has a past end point on A, or else is past inextendible in M.*



Proof.



Choose $\{p_n\} \subset I^+(A)$ such that $p_n \to p$, and let γ_n be a past directed timelike curve from p_n to A. By Ascoli, and passing to a subsequence, $\{\gamma_n\}$ converges to a past directed causal curve $\gamma \subset \partial I^+(A)$ from p. Since γ is both causal and achronal, it must be a null geodesic.

Each γ_n is past inextendible in $M \setminus A$, and hence so is γ. Thus γ either has a past end point on A or is past inextendible in M.

C^0 null hypersurfaces.

Thus sets of the form

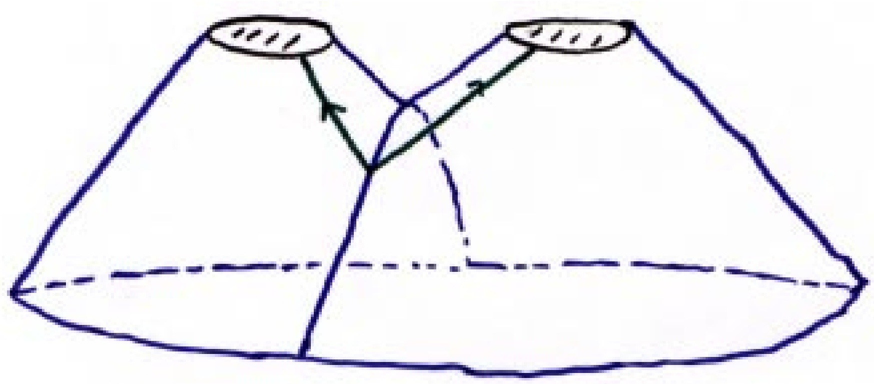
$$S = \partial I^+(A) \setminus A, \quad \text{resp.,} \quad S = \partial I^-(A) \setminus A\,,$$

with A closed, are achronal C^0 hypersurfaces, ruled by null geodesics which are past, resp. future, inextendible in S.

Def. A C^0 *future* null hypersurface is a locally achronal C^0 hypersurface S, which is ruled by null geodesics that are *future* inextendible in S.

Ex. $S = \partial I^-(A) \setminus A$, $A \subset M$ closed.

Ex. $M =$ Minkowski 3-space, $A =$ two disjoint spacelike disks in $t = 0$. Then $S = \partial I^-(A) \setminus A$ is a C^0 future null hypersurface in M



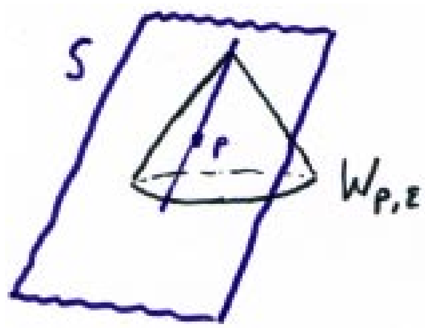
A C^0 *past* null hypersurface is defined time-dually: it is ruled by null geodesics that are *past* inextendible within the hypersurface.

Mean curvature inequalities for C^0 null hypersurfaces.

C^0 null hypersurfaces do not have null mean curvature in the classical sense, but may obey null mean curvature inequalities in a *support sense*.

Def. Let S be a C^0 future null hypersurface. S has null mean curvature $\theta \geq 0$ *in the support sense* provided $\forall p \in S$, and $\forall \epsilon > 0$, there exists a smooth (C^2) null hypersurface $W_{p,\epsilon}$ such that

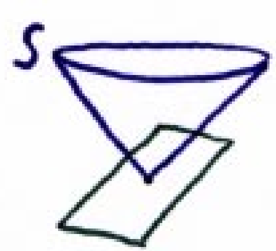
- $W_{p,\epsilon}$ is a past support hypersurface for S at p.
- $\theta_{p,\epsilon}(p) \geq -\epsilon$.



(For this definition, it is assumed that the null vectors have been uniformly scaled, e.g., have unit length wrt a background Riemannian metric.)

Note: If S is smooth then $\theta \geq 0$ in the usual sense.

Ex. M = Minkowski space, $S = \partial I^+(p)$. S is a C^0 future null hypersurface having $\theta \geq 0$ in the support sense.



If S is a C^0 *past* null hypersurface, one defines $\theta \leq 0$ *in a support sense* in an analogous manner in terms of *future* support hypersurfaces.

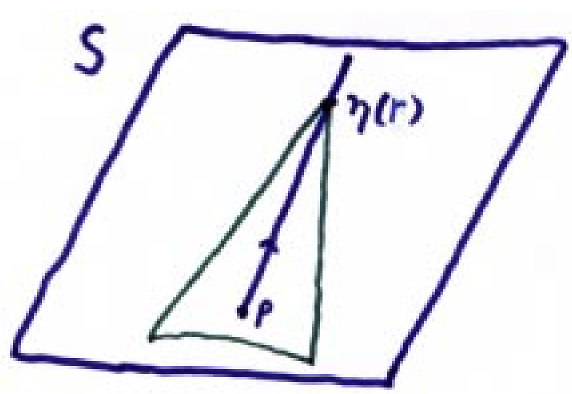
Prop. *Let S be a C^0 future null hypersurface in M. Suppose,*

- *M obeys the null energy condition.*
- *The null generators of S are future geodesically complete.*

Then $\theta \geq 0$ in the support sense.

Proof: WLOG, may assume S is achronal. Given $p \in S$, let $\eta : [0, \infty) \to S \subset M$, $s \to \eta(s)$, be a null generator of S from $p = \eta(0)$.

For any $r > 0$, consider small pencil of past directed null geodesics from $\eta(r)$. Forms a smooth (caustic free) null hypersurface $W_{p,r}$ containing $\eta|_{[0,r]}$, which is a lower support hypersurface for S at p.



Let $\theta = \theta(s)$, $0 \leq s \leq r$, be the null mean curvature of $W_{p,r}$ along $\eta|_{[0,r]}$.

By Raychaudhuri and NEC we have,

$$\frac{d\theta}{ds} \quad \leq \quad -\frac{1}{n-1}\theta^2,$$

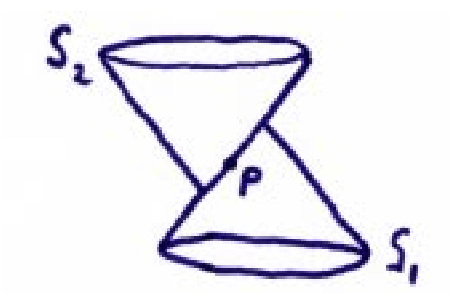
Together with $\theta(r) = -\infty$ gives,

$$\theta(0) \geq -\frac{n-1}{r}$$

Maximum principle for C^0 null hypersurfaces.

Theorem. *Suppose*

- *S_1 is a C^0 future null hypersurface, and S_2 is a C^0 past null hypersurface in M.*
- *S_1 , S_2 meet at $p \in M$, with S_2 to the future side of S_1 near p.*



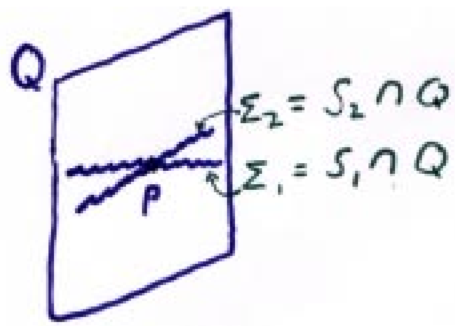
- *$\theta_2 \leq 0 \leq \theta_1$ in the support sense.*

Then S_1 and S_2 coincide near p, and form a smooth *null hypersurface with $\theta = 0$.*

Comments on the proof:

Although there are some technical issues, the proof proceeds essentially as in the smooth case.

Can show p is an interior point of a null generator common to both S_1 and S_2 near p. As before, intersect S_1 and S_2 with a timelike hypersurface Q through p, transverse to this generator.



Σ_1 and Σ_2 will be C^0 *spacelike* hypersurfaces in Q, with Σ_2 to the future of Σ_1 near p.

Can express Σ_1 and Σ_2 as graphs over a fixed hypersurface in Q,

$$\Sigma_1 = \text{graph}\,(u_1)\,, \quad \Sigma_2 = \text{graph}\,(u_2)$$

One has:

- $u_1 \le u_2$, and $u_1(p) = u_2(p)$.
- $\theta(u_2) \le 0 \le \theta(u_1)$ in the support sense.

Need a suitable weak version of the strong maximum principle: Andersson, Howard, G. ('98, Comm. Pure Appl. Math.)

For further details, see: G., Ann. Henri Poincaré **1** (2000) 543.

The Null Splitting Theorem

Lines in Spacetimes.

A *timelike line* is an inextendible timelike geodesic each segment of which is maximal.

The standard Lorentzian splitting theorem describes the rigidity of spacetimes containing a timelike line:

Theorem. *Suppose*

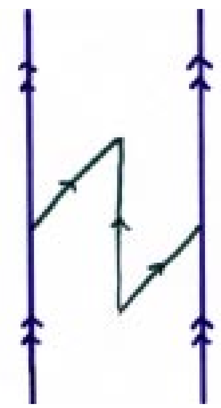
- M *is timelike geodesically complete.*
- M *obeys the strong energy condition,* $\text{Ric}(X, X) \geq 0$, *for timelike* X.
- M *has a timelike line.*

Then M *splits isometrically along the line, i.e.,* (M, g) *is isometric to* $(\mathbb{R} \times V, -dt^2 \oplus h)$, *where* (V, h) *is a complete Riemannian manifold.*

Comment:

- Precise analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry.
- Recall, posed as a problem by Yau in the early 80's as an approach to removing the genericity assumptions in the Hawking-Penrose singularity theorems.

A null line in a spacetime M is an inextendible null geodesic which is *achronal*, i.e. no two points can be joined by a timelike curve. (Thus each segment of a null line is maximal.)



- Global condition.
- Null lines arise naturally in causal arguments: E.g., recall sets of the form,

$$\partial I^{\pm}(A) \setminus A\,, \qquad A \text{ closed}$$

  are ruled by null geodesics which must be achronal.
- Null lines have arisen in many situations, e.g., the Hawking-Penrose singularity theorems, topological censorship, Penrose-Sorkin-Woolgar approach to positive mass, and related results of Gao-Wald on gravitational time delay, etc.
- Examples: Minkowski space, de Sitter, anti-de Sitter, Schwarzschild

One expects some rigidity in spacetimes which contain a null line and which obey the null energy condition.

The NEC tends to focus congruences of null geodesics, which can lead to the occurence of null conjugate points. A null geodesic containing a pair of null conjugate points can't be achronal.

Theorem (*Null Splitting Theorem*). *Suppose,*

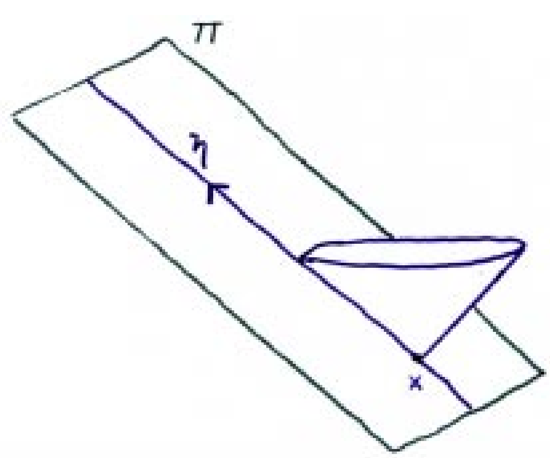
- M *is null geodesically complete.*
- M *obeys the null energy condition,* Ric $(X, X) \geq 0$ *for all null* X.
- M *contains a null line* η.

Then η *is contained in a smooth closed achronal totally geodesic null hypersurface* S.

(1) <u>*Ex.*</u> Minkowski space - each null geodesic is contained in a unique null hyperplane.

(2) The "splitting" is in S: $B = 0 \iff \theta = \sigma = 0 \iff$ metric h on TS/K is invariant under flow generated by K.

(3) The proof is an application of the maximum principle for C^0 null hypersurfaces. To motivate, consider the situation in Minkowski space:



$$\Pi = \lim_{x \to -\infty} \partial I^+(x) = \partial I^+(\eta)$$

$$= \lim_{y \to \infty} \partial I^-(y) = \partial I^-(\eta)$$

Proof: Set,

$$S_+ = \partial I^+(\eta)\,, \qquad S_- = \partial I^-(\eta)$$

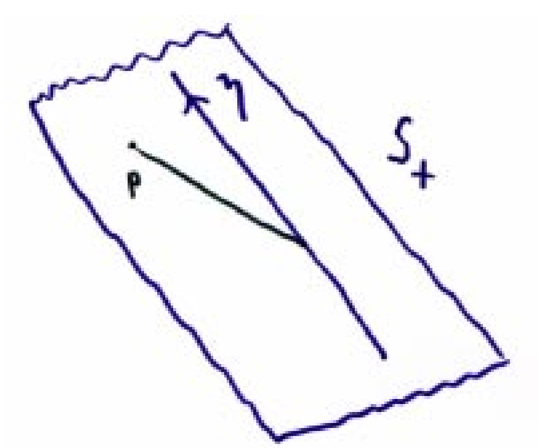
Since η is achronal, $\eta \subset S_+ \cap S_-$.

Claim. S_+ is a C^0 past null hypersurface whose null generators are past inextendible in M. (Similarly for S_-.)

Pf: As an achronal boundary, S_+ is an achronal C^0 hypersurface.

Now, for simplicity assume M is strongly causal. Then η is closed as a subset of M.

By property of achronal boundaries, each point $p \in S_+ \setminus \eta$ is on a null geodesic which is either past inextendible in M or else has past endpoint on η. The latter is impossible:



This violates the achronality of S_+.

Claim. The null mean curvature of S_- and of S_+ satisfy,

$$\theta_+ \le 0 \le \theta_- \quad \text{in the support sense.}$$

<u>*Pf:*</u> By the completeness assumption, and the previous claim, the generators of S_+ are past complete, and the generators of S_- are future complete. Thus, the claim follows from a previous proposition.

At each point of the intersection $p \in S_+ \cap S_-$, S_+ lies locally to the future of S_-. Thus be the maximum principle $S+$ and S_- agree near p, and form a smooth null hypersurface with vanishing null mean curvature.

It follows that $S_+ \cap S_-$ is both open and closed in S_+ and in S_-. Thus,

$$S_+ = S_+ \cap S_- = S_-\,,$$

and $S = S_+ = S_-$ is a smooth null hypersurface with $\theta = 0$.

Raychaudhuri's equation,

$$\frac{d\theta}{ds} = -\mathrm{Ric}\,(\eta', \eta') - \sigma^2 - \frac{1}{n-1}\theta^2$$

and the NEC now imply that S is totally geodesic.

<u>*Note:*</u> With regard to the completeness assumption, the proof only requires that the generators of $\partial I^+(\eta)$ be past complete and the generators of $\partial I^-(\eta)$ be future compete.