Einstein equations, conformal structure, and the asymptotic behaviour of space-time

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Null cone structure:
main tool for exploration of gravitational fields in the large.

Differential topology and differential geometry

null cone structure $\leftrightarrow$ *causal structure*

causal sets, boundaries of causal sets, horizons, ...,

Einstein's field equations $\tilde{R}_{\mu\nu} = \lambda\,\tilde{g}_{\mu\nu} + \kappa\,(T_{\mu\nu} - \frac{1}{2}T\,\tilde{g}_{\mu\nu})$

null cone structure $\leftrightarrow$ *characteristics*

positivity conditions, Raychaudhuri equation ...: 'singularities'
formal expansions, analytical control, numerical calculations, ...

Differential geometry and field equations

null cone structure $\leftrightarrow$ *conformal structure*

substructure which is invariant under conformal rescalings

$$\tilde{g} \rightarrow g = \Omega^2\,\tilde{g}, \quad \Omega > 0.$$

Ω **small: large 'physical' space-time regions are small w.r.t.** g**-relations.**

If $\Omega \rightarrow 0$ **at 'infinity' the** g**-relations may remain finite on a space-time of infinite 'physical' extent**

Einstein's equations are not conformally covariant ($n = dim\,M$)

$$R_{\nu\rho}[g] = \tilde{R}_{\nu\rho}[\tilde{g}] - \frac{n-2}{\Omega}\,\nabla_\nu\,\nabla_\rho\,\Omega - g_{\nu\rho}\,\left(\frac{1}{\Omega}\,\nabla_\lambda\,\nabla^\lambda\,\Omega - \frac{n-1}{\Omega^2}\,\nabla_\lambda\,\Omega\,\nabla^\lambda\,\Omega\right)$$

principal part retained, right hand side degenerates as $\Omega \rightarrow 0$**.**

Plan

Lecture 1 **Formal properties of the field equations, the conformal field equations.**

Lecture 2 **The Penrose proposal, problems and results.**

Lecture 3 **Results on the asymptotic conformal structure of asymptotically flat vacuum solutions with vanishing cosmological constant.**

Formal properties of the field equations, the conformal field equations.

Conformal geometry:

A conformal rescaling

$$\tilde{g} \to g = \Omega^2 \, \tilde{g}$$

implies transitions $\tilde{\nabla} \to \nabla$ of the Levi-Civita connection and of the Christoffel symbols

$$\tilde{\Gamma}_\mu{}^\rho{}_\nu \to \Gamma_\mu{}^\rho{}_\nu = \tilde{\Gamma}_\mu{}^\rho{}_\nu + S(\Omega^{-1} \, d\,\Omega)_\mu{}^\rho{}_\nu$$

where we write, for any 1-form f,

$$S(f)_\mu{}^\rho{}_\nu \equiv \delta^\rho{}_\mu \, f_\nu + \delta^\rho{}_\nu \, f_\mu - g_{\mu\nu} \, g^{\rho\lambda} \, f_\lambda.$$

Decomposition of curvature tensor $(dim\ M = n \geq 4)$

$$R^\mu{}_{\nu\lambda\rho} = 2 \, \{ g^\mu{}_{[\lambda} \, L_{\rho]\nu} - \, g_{\nu[\lambda} \, L_{\rho]}{}^\mu \} + C^\mu{}_{\nu\lambda\rho},$$

conformal Weyl tensor $C^\mu{}_{\nu\lambda\rho}$, **traces** $L_{\mu\nu} \equiv \dfrac{1}{n-2} \left(R_{\mu\nu} - \dfrac{1}{2\,(n-1)} \, R \, g_{\mu\nu} \right).$

Bianchi and once contracted Bianchi identity

$$\nabla_{[\delta} \, C^\mu{}_{|\nu|\lambda\rho]} = 2 \, (g_{\nu[\lambda} \, \nabla_\delta \, L_{\rho]}{}^\mu - g^\mu{}_{[\lambda} \, \nabla_\delta \, L_{\rho]\nu})$$

$$\nabla_\mu \, C^\mu{}_{\nu\lambda\rho} = (n-3) \, (\nabla_\lambda \, L_{\rho\nu} - \nabla_\rho \, L_{\lambda\nu})$$

Conformal invariance

$$C^\mu{}_{\nu\lambda\rho}[g] = \tilde{C}^\mu{}_{\nu\lambda\rho}[\tilde{g}]$$

Conformal non-covariance

$$L_{\mu\nu}[g] = \tilde{L}_{\mu\nu}[\tilde{g}] - \frac{1}{\Omega} \, \nabla_\mu \, \nabla_\nu \, \Omega + \frac{1}{2\,\Omega^2} \, g_{\mu\nu} \, \nabla_\rho \Omega \, \nabla^\rho \Omega$$

Conformal covariance

$$\nabla_\mu \, (\Omega^{3-n} \, C^\mu{}_{\nu\lambda\rho}) = \Omega^{3-n} \, \left\{ \nabla_\mu \, C^\mu{}_{\nu\lambda\rho} - (n-3) \, \Omega^{n-4} \, \nabla_\mu \, \Omega \, (\Omega^{3-n} \, C^\mu{}_{\nu\lambda\rho}) \right\}$$

$$= \Omega^{3-n} \, \tilde{\nabla}_\mu \, \tilde{C}^\mu{}_{\nu\lambda\rho}$$

The metric conformal field equations I.

Assume that $\tilde{g}$ satisfies Einstein's vacuum field equations $\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu}$ and thus

$$\tilde{L}_{\mu\nu} = \frac{\lambda}{2\,(n-1)}\, \tilde{g}_{\mu\nu} \quad \textbf{and} \quad \tilde{\nabla}_\mu\, C^\mu{}_{\nu\lambda\rho} = 0.$$

Exploit available conformal covariance to derive a regular system of equations for Ω and the conformal metric $g = \Omega^2\, \tilde{g}$. Set

$$d^\mu{}_{\nu\lambda\rho} \equiv \Omega^{3-n}\, C^\mu{}_{\nu\lambda\rho}$$

The equations above imply for the metric and the conformal curvature fields

$$R^\mu{}_{\nu\lambda\rho} = 2\,\{g^\mu{}_{[\lambda}\, L_{\rho]\nu} - \; g_{\nu[\lambda}\, L_{\rho]}{}^\mu\} + \Omega^{n-3}\, d^\mu{}_{\nu\lambda\rho}$$

$$\nabla_\lambda\, L_{\rho\nu} - \nabla_\rho\, L_{\lambda\nu} = \Omega^{n-4}\, \nabla_\mu\, \Omega\, d^\mu{}_{\nu\lambda\rho}$$

$$\nabla_\mu\, d^\mu{}_{\nu\lambda\rho} = 0 \quad \textbf{(Bianchi equation)}$$

To obtain equations controlling the evolution of the conformal factor use the trace-free part and the trace of the conformal Ricci tensor

$$\nabla_\mu\, \nabla_\nu\, \Omega = -\Omega\, L_{\mu\nu} + s\, g_{\mu\nu},$$

$$\frac{\lambda}{n-1} = 2\,\Omega\, s - \nabla_\rho\, \Omega\, \nabla^\rho\, \Omega.$$

with scalar field

$$s \equiv \frac{1}{n}\, \nabla_\lambda\, \nabla^\lambda\, \Omega + \frac{1}{2\,n\,(n-1)}\, R\, \Omega$$

The equations above imply the integrability condition

$$\nabla_\mu\, s = -\nabla^\nu\, \Omega\, L_{\nu\mu}$$

The metric conformal field equations II.

Unknown

$$u = (g, \Omega, s, L_{\mu\nu}, d^{\mu}{}_{\nu\lambda\rho})$$

Metric conformal field equations

$$R^{\mu}{}_{\nu\lambda\rho} = 2\,\{g^{\mu}{}_{[\lambda}\,L_{\rho]\nu} - \,g_{\nu[\lambda}\,L_{\rho]}{}^{\mu}\} + \Omega^{n-3}\,d^{\mu}{}_{\nu\lambda\rho}$$

$$\nabla_{\mu}\,\nabla_{\nu}\,\Omega = -\Omega\,L_{\mu\nu} + s\,g_{\mu\nu}$$

$$\nabla_{\mu}\,s = -\nabla^{\nu}\,\Omega\,L_{\nu\mu}$$

$$\nabla_{\lambda}\,L_{\rho\nu} - \nabla_{\rho}\,L_{\lambda\nu} = \Omega^{n-4}\,\nabla_{\mu}\,\Omega\,d^{\mu}{}_{\nu\lambda\rho}$$

$$\nabla_{\mu}\,d^{\mu}{}_{\nu\lambda\rho} = 0$$

$$\frac{\lambda}{n-1} = 2\,\Omega\,s - \nabla_{\rho}\,\Omega\,\nabla^{\rho}\,\Omega$$

On a connected set the last equation follows if it holds at one point.

Conformal gauge freedom: equations are invariant under

$$g \longrightarrow g^{*} = \theta^{\frac{4}{n-2}}\,g, \quad \Omega \longrightarrow \Omega^{*} = \theta^{\frac{2}{n-2}}\,\Omega, \quad d^{\mu}{}_{\nu\lambda\rho} \longrightarrow d^{*\mu}{}_{\nu\lambda\rho} = \theta^{-\frac{2}{n-2}}\,d^{\mu}{}_{\nu\lambda\rho},$$

$$s \longrightarrow s^{*} = \theta^{-\frac{2}{n-2}}\,s + \frac{2}{n\,(n-2)}\,\theta^{-\frac{2\,(n-1)}{n-2}}\,(2\,\Omega\,\nabla_{\mu}\theta\,\nabla^{\mu}\theta + n\,\theta\,\nabla_{\mu}\theta\,\nabla^{\mu}\Omega),$$

$$L_{\mu\nu} \longrightarrow L^{*}_{\mu\nu} = L_{\mu\nu}$$

$$-\frac{2}{(n-2)^2}\,\theta^{-2}\,((n-2)\,\theta\,\nabla_{\mu}\,\nabla_{\nu}\,\theta - n\,\nabla_{\mu}\,\theta\,\nabla_{\nu}\,\theta + g_{\mu\nu}\,\nabla_{\rho}\theta\,\nabla^{\rho}\theta).$$

In particular

$$L_{\mu}{}^{\mu} \longrightarrow L^{*}_{\mu}{}^{\mu} = \theta^{-\frac{n+2}{n-2}}\left(\theta\,L_{\mu}{}^{\mu} - \frac{2}{n-2}\,\nabla_{\mu}\,\nabla^{\mu}\,\theta\right).$$

Suggests to remove the gauge freedom by prescribing the function $R = 2\,(n-1)\,L_{\mu}{}^{\mu}$ **as the conformal gauge source function.**

Then Ω is determined implicitly by a wave equation.

The Bianchi equation I.

The vacuum Bianchi identity

$$\tilde{\nabla}_\delta C^\mu{}_{\nu\lambda\rho} + \tilde{\nabla}_\rho C^\mu{}_{\nu\delta\lambda} + \tilde{\nabla}_\lambda C^\mu{}_{\nu\rho\delta} = 0$$

implies in $n \geq 4$ **dimensions the system of wave equations**

$$\tilde{\nabla}_\delta \tilde{\nabla}^\delta C^\mu{}_{\nu\tau\rho} = 2\,\{C^\mu{}_{\pi\delta\rho}\, C^\pi{}_{\nu\tau}{}^\delta + C^\mu{}_{\pi\tau\delta}\, C^\pi{}_{\nu\rho}{}^\delta + C^\mu{}_{\nu\pi\delta}\, C^\pi{}_{\tau\rho}{}^\delta\} + 2\,\lambda\, C^\mu{}_{\nu\tau\rho}$$

and is thus always 'intrinsically hyperbolic'.

$n = 4$: **Bianchi identity ⇔ contracted Bianchi identity**

$$\tilde{\epsilon}_\pi{}^{\delta\tau\rho} \tilde{\nabla}_\delta C^\mu{}_{\nu\tau\rho} = 2\,\tilde{\nabla}_\delta C^{*\mu}{}_{\nu\pi}{}^\delta = 2\,\tilde{\nabla}_\delta {}^*C^\mu{}_{\nu\pi}{}^\delta = -\,\tilde{\epsilon}^\mu{}_\nu{}^{\tau\rho}\, \tilde{\nabla}_\delta C^\delta{}_{\pi\tau\rho}$$

thus: Bianchi equation $\nabla_\delta\, d^\delta{}_{\pi\tau\rho} = 0$ **also 'intrinsically hyperbolic'.**

Conformal Weyl tensor: $u = \frac{1}{12}\, n^2\,(n^2 - 1) - \frac{1}{2}\, n\,(n+1)$ **independent components,**

Contracted Bianchi id.: $e = \frac{1}{3}\, n\,(n-2)\,(n+2)$ **independent equations,**

$c = \frac{1}{2}\, n\,(n-1)$ **constraints**

($\tilde{\nabla}_\delta\, C^\delta{}_{0\tau\rho} = 0$ **in Gauss coordinates)**

$n \geq 5$: $e - c < u$ **evolution equations in the contracted Bianchi identity.**

$\nabla_\delta\, d^\delta{}_{\pi\tau\rho} = 0$ **does not provide hyperbolic evolution equations.**

∃ **regular, hyperbolic conformal evolution equations if** $n \geq 5$ **?**

∃ **differences in asymptotic behaviour between** $n = 4$ **and** $n \geq 5$ **?**

Assume from now on: $n = 4$

The Bianchi equation II.

There are various ways to exhibit the hyperbolicity of the Bianchi equation.

If the function t defines a space-like foliation, we write

$$g = (N\,d\,t)^2 - h_{ij}\,(S^i\,d\,t + d\,x^i)\,(S^j\,d\,t + d\,x^j)$$

$$n^\mu = \frac{1}{N}\,(\delta^\mu{}_0 - S^\mu), \quad a_\mu = \frac{1}{N}\,D_\mu\,N, \quad h^\mu{}_\nu = g^\mu{}_\nu - n^\mu\,n_\nu, \quad \chi_{\mu\nu} = \frac{1}{2}\,\mathcal{L}_n\,h_{\mu\nu}$$

$$C_{\mu\nu\lambda\rho} = 2\,\{l_{\mu[\lambda}\,E_{\rho]\nu} - l_{\nu[\lambda}\,E_{\rho]\mu} - n_{[\lambda}\,B_{\rho]\tau}\,\epsilon^\tau{}_{\mu\nu} - n_{[\mu}\,B_{\nu]\tau}\,\epsilon^\tau{}_{\lambda\rho}\}$$

and denote by D the h Levi-Civita operator on $\{t = const.\}$,

we obtain from the Bianchi equations the constraints

$$D^i\,E_{ij} - B_{ik}\,\chi^i{}_l\,\epsilon^{kl}{}_j = 0$$

$$D^i\,B_{ij} + E_{ik}\,\chi^i{}_l\,\epsilon^{kl}{}_j = 0,$$

which are uniquely determined by the hypersurface $\{t = const.\}$

and propagation equations

$$\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,E_{ij} + D_k\,B_{l(i}\,\epsilon_{j)}{}^{kl} - 3\,E^k{}_{(i}\,\chi_{j)k} + \chi_k{}^k\,E_{ij} - \epsilon_i{}^{kl}\,E_{km}\,\chi_{ln}\,\epsilon_j{}^{mn}$$
$$+2\,a_k\,B_{l(i}\,\epsilon_{j)}{}^{kl} = 0,$$

$$\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,B_{ij} - D_k\,E_{l(i}\,\epsilon_{j)}{}^{kl} - 3\,B^k{}_{(i}\,\chi_{j)k} + \chi_k{}^k\,B_{ij} - \epsilon_i{}^{kl}\,B_{km}\,\chi_{ln}\,\epsilon_j{}^{mn}$$
$$-2\,a_k\,E_{l(i}\,\epsilon_{j)}{}^{kl} = 0,$$

which are not unique.

The latter imply the manifestly symmetric hyperbolic system

$$h^{k(i}h^{j)l}\,\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,E_{ij} + \frac{1}{2}\,(\epsilon^{ij(k}\,h^{l)m} + \epsilon^{im(k}\,h^{l)j})\,D_i\,B_{jm} + ... = 0,$$

$$h^{k(i}h^{j)l}\,\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,B_{ij} - \frac{1}{2}\,(\epsilon^{ij(k}\,h^{l)m} + \epsilon^{im(k}\,h^{l)j})\,D_i\,E_{jm} + ... = 0.$$

The Bianchi equation III.

In the spin frame formalism $d^{\mu}{}_{\nu\lambda\rho}$ is represented by a symmetric spinor field ϕ_{abcd}

The Bianchi equation reads

$$0 = \Lambda_{abca'} \equiv \epsilon^{ef}\,\nabla_{a'f}\,\phi_{abce}$$

with $a, b, c, \ldots, a', b', c' \ldots = 0, 1$, and $\epsilon^{ab} = \epsilon^{[ab]}$, $\epsilon^{01} = 1$ $\epsilon^{ab}\,\epsilon_{ac} = \delta^b{}_c$, $\nabla_{aa'}$ the covariant derivative in the direction of $e_{aa'}$, where

$$e_{00'} = 1/\sqrt{2}\,(e_0 + e_3), \quad e_{11'} = 1/\sqrt{2}\,(e_0 - e_3),$$

$$e_{01'} = 1/\sqrt{2}\,(e_1 + i\,e_2), \quad e_{10'} = 1/\sqrt{2}\,(e_1 - i\,e_2)$$

with a frame field e_k satisfying $g(e_i, e_k) = \eta_{ik}$ such that $g(e_{aa'}, e_{bb'}) = \epsilon_{ab}\,\epsilon_{a'b'}$.

With $\tau^{aa'} = \delta_0{}^a\,\delta_{0'}{}^{a'} + \delta_1{}^a\,\delta_{1'}{}^{a'}$, $P = \tau^{aa'}\,\nabla_{aa'} = \sqrt{2}\,\nabla_{e_0}$, $\mathcal{D}_{ab} = \tau_{(a}{}^{a'}\,\nabla_{b)a'} \perp P$:

$$\Lambda_{abca'} = 0 \quad \textbf{iff} \quad \Lambda_{abcd} \equiv \Lambda_{abca'}\,\tau_d{}^{a'} = 0 \;\; \textbf{iff}$$

$$0 = \Lambda_{abf}{}^{f} = \mathcal{D}^{ef}\,\phi_{abef} \quad \textbf{(constraints)}, \quad 0 = \Lambda_{(abcd)} = -\frac{1}{2}\,P\,\phi_{abcd} + \mathcal{D}_{(d}{}^{f}\,\phi_{abc)f}$$

with symmetric hyperbolic version (10 real equations)

$$-\binom{4}{a+b+c+d}\,\Lambda_{(abcd)} = 0.$$

Further symmetric hyperbolic systems:

$$0 = 2\,a\,\Lambda_{(0001')},$$

$$0 = (c-d)\,\Lambda_{(0011')} - 2\,a\,\Lambda_{(0000')},$$

$$0 = (c+d)\,\Lambda_{(0111')} - (c-d)\,\Lambda_{(0010')},$$

$$0 = 2\,e\,\Lambda_{(1111')} - (c+d)\,\Lambda_{(0110')},$$

$$0 = -2\,e\,\Lambda_{(1110')},$$

with $a, c, e > 0$, $-(2\,e + c) < d < 2\,a + c$.

The conformal constraints I.

S **space-like hypersurface,** $\{e_k\}_{k=0,1,2,3}$ **frame near** S**,** $g(e_j, e_k) = \eta_{jk}$**,** $e_0 \perp S$
Write tensors in this frame, assume $a, b, c, .. = 1, 2, 3$ **and sum convention.**
Denote by D **the Levi-Civita connection of the induced metric** h**, set**

$$h_{ab} = h(e_a, e_b), \quad \epsilon_{abc} = \epsilon_{[abc]}, \quad \textbf{with } \epsilon_{123} = 1$$

$$\Sigma = e_0(\Omega), \quad \chi_{ab} = g(\nabla_{c_a} e_0, c_b), \quad L_a = L_{a0}, \quad d_{ab} = d_{a0b0}, \quad d^*_{ab} = d^*_{a0b0}$$

such that

$$d_{ab} = d_{(ab)}, \quad d_a{}^a = 0, \quad d^*_{ab} = d^*_{(ab)}, \quad d^*_a{}^a = 0, \quad d_{abcd} = 2\{h_{a[c}d_{d]b} + h_{b[d}d_{c]a}\}$$

Interior equations implied by metric conformal field equations on S**:**

$$^3R_{ab} = \chi_c{}^c \chi_{ab} - \chi_{ca}\chi_b{}^c + \Omega\, d_{ab} + L_{ab} + h_{ab} L_c{}^c$$

$$D_b \chi_{ca} - D_c \chi_{ba} = \Omega\, d^*_{ae} \epsilon^e{}_{bc} + 2\, h_{a[b} L_{c]}$$

$$D_a D_b \Omega = -\Sigma\, \chi_{ab} - \Omega\, L_{ab} + s\, h_{ab}$$

$$D_a \Sigma = \chi_a{}^c D_c \Omega - \Omega\, L_a$$

$$D_a s = -D^b \Omega\, L_{ba} - \Sigma\, L_a$$

$$D_a L_{bc} - D_b L_{ac} = D^e \Omega\, d_{ecab} - \Sigma\, d^*_{ce} \epsilon^e{}_{ab} + 2\, \chi_{c[a} L_{b]}$$

$$D_a L_b - D_b L_a = D^f \Omega\, d^*_{fe} \epsilon^e{}_{ab} + 2\, \chi_{[a}{}^c L_{b]c}$$

$$D^a d^*_{ab} = -\chi^c{}_e\, d_{cf}\, \epsilon_b{}^{ef}$$

$$D^a d_{ab} = \chi^c{}_e\, d^*_{cf}\, \epsilon_b{}^{ef}$$

$$\lambda = 6\,\Omega\, s - 3\,\Sigma^2 - 3 D_a \Omega\, D^a \Omega$$

Conformal constraints complicated because of integrability conditions and conformal rescaling. The standard vacuum constraints read

$$^3R = (\chi_c{}^c)^2 - \chi_{cd}\,\chi^{cd} + 2\,\lambda, \qquad D_a \chi_c{}^a - D_c \chi_a{}^a = 0.$$

The conformal constraints II.

Standard vacuum constraints

$$^3R = (\chi_c{}^c)^2 - \chi_{cd}\,\chi^{cd} + 2\,\lambda, \qquad D_a\chi_c{}^a - D_c\chi_a{}^a = 0.$$

The conformal field equations include integrability conditions, thus even for $\Omega \equiv 1$ we find the more complicated equations

$$^3R_{ab} = \chi_c{}^c\,\chi_{ab} - \chi_{ca}\,\chi_b{}^c + 2/3\,\lambda\,h_{ab} + d_{ab}$$

$$D_b\chi_{ca} - D_c\chi_{ba} = d^*_{ae}\,\epsilon^e{}_{bc}$$

$$D^a d^*_{ab} = -\chi^c{}_e\,d_{cf}\,\epsilon_b{}^{ef}$$

$$D^a d_{ab} = \chi^c{}_e\,d^*_{cf}\,\epsilon_b{}^{ef}$$

Standard strategy to provide solutions:

Prescribe free data: conformal metric $\hat{h}_{ab}$, 'trial' $\hat{\chi}_{ab}$,

Solve linear resp. semi-linear elliptic equations to obtain solution to the standard constraints (Lichnerowicz, York, Choquet-Bruhat, ...)

Use first subsystem above to define d_{ab} and d^*_{ab}.
The other equations will then be satisfied.

Procedure works for conformal constraints similarly.
Delicate: behaviour near $\Omega = 0$.

Different strategy:

Prescribe free data: 'trial' $\hat{d}_{ab}$, $\hat{d}^*_{ab}$

Read the system as a 3-dimensional Einstein equation with source fields

$$d_{ab} = D_a\,X_b + D_b\,X_a - \frac{2}{3}\,h_{ab}\,D_c\,X^c + \hat{d}_{ab}$$

$$d^*_{ab} = D_a\,X^*_b + D_b\,X^*_a - \frac{2}{3}\,h_{ab}\,D_c\,X^{*c} + \hat{d}^*_{ab}$$

Solve quasi-linear elliptic system for h_{ab}, χ_{ab}, X_a, X^*_a.

First task: formulate well posed elliptic boundary value problems.

A. Butscher (2002): Stability of asymptotically euclidean vacuum solution $h_{ab} = -\delta_{ab}$, $\chi_{ab} = 0$, $d_{ab} = 0$, $d^*_{ab} = 0$.

Gauge conditions, hyperbolicity, and local evolution.

The conformal field equations pose the same problems and offer the same flexibility as Einstein's equations in their standard form.

$\exists$ various ways to handle conformal, coordinate, and frame gauge freedom:
Implicit wave equations for conformal factor, coordinates, frame field, or geometrical gauge with explicit algebraic conditions on unknown, ...

All techniques developed for Einstein's field equations may be tried:
Writing formally $\kappa\, T_{\mu\nu} \equiv 2\,(L_{\mu\nu} - L\, g_{\mu\nu})$, the metric conformal field equations take the form of

'Einstein equations with matter fields'

$$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} = \kappa\, T_{\mu\nu}$$

$$\nabla_\mu\, \nabla_\nu \Omega = -\frac{\kappa}{2}\,(T_{\mu\nu} - \frac{1}{3}\, T\, g_{\mu\nu})\, \Omega + s\, g_{\mu\nu}$$

$$\nabla_\mu\, s = -\frac{\kappa}{2}\,(T_{\mu\nu} - \frac{1}{3}\, T\, g_{\mu\nu})\, \nabla^\nu\, \Omega$$

$$\kappa\,(\nabla_{[\lambda} T_{\rho]\nu} - \frac{1}{3}\, \nabla_{[\lambda} T\, g_{\rho]\nu}) = \nabla_\mu \Omega\, d^\mu{}_{\nu\lambda\rho}$$

$$\nabla_\mu\, d^\mu{}_{\nu\lambda\rho} = 0$$

(ensure geometrical role of $T_{\mu\nu}$ and $d^\mu{}_{\nu\lambda\rho}$)

$\exists$ hyperbolic systems in the tensor, frame, and spin frame formalism.

$\exists$ various ways to formulate well posed local initial value problems for the metric conformal field equations (preserving constraints and gauge conditions)

Specific for the conformal field equations:

Various global and semi-global geometric problems for Einstein's field equations can be/have been recast as well posed
−characteristics initial value problems,
−Cauchy problems,
−initial boundary value problems
for the conformal field equations, involving finite domains where Ω changes sign.

The general conformal field equations I.

Besides conformal rescalings $\tilde{g} \to g = \Theta^2\, \tilde{g}$
consider also Weyl connections $\hat{\nabla} = \nabla_g + S(f)$ **with arbitrary 1-form** f**,**
torsion free, respect conformal class $\mathcal{C}_g$**:** $\hat{\nabla}_\rho\, g_{\mu\nu} = -2\, f_\rho\, g_{\mu\nu}$**.**

Representation of 1-form depends on conformal scaling of metric
$\nabla_g = \nabla_{\tilde{g}} + S(\Theta^{-1}\, \nabla\Theta)$**,** $\hat{\nabla} = \nabla_{\tilde{g}} + S(\Theta^{-1}\, d)$ **with** $d = \Theta\, f + \nabla\Theta$**.**

Write fields in frame $\{e_k\}_{k=0,\ldots,3}$ **with** $g(e_i, e_k) = \eta_{ik}$**. Set**

$$\hat{\nabla}_i e_j \equiv \hat{\nabla}_{e_i} e_j = \hat{\Gamma}_i{}^k{}_j\, e_k, \quad d^i{}_{jkl} = \Theta^{-1}\, C^i{}_{jkl},$$

$$\hat{L}_{jk} = 1/2\, \hat{R}_{(jk)} - 1/4\, \hat{R}_{[jk]} - 1/12\, \hat{R}\, g_{jk}$$

The vacuum field equation $Ric[\tilde{g}] = \lambda\, \tilde{g}$ **then imply for**

$$u = (e^\mu{}_k,\ \hat{\Gamma}_i{}^j{}_k,\ \hat{L}_{jk},\ d^i{}_{jkl}),$$

the 'conformal equations based on Weyl connections'

$$[e_p, e_q] = (\hat{\Gamma}_p{}^l{}_q - \hat{\Gamma}_q{}^l{}_p)\, e_l,$$

$$e_p(\hat{\Gamma}_q{}^i{}_j) - e_q(\hat{\Gamma}_p{}^i{}_j) - \hat{\Gamma}_k{}^i{}_j(\hat{\Gamma}_p{}^k{}_q - \hat{\Gamma}_q{}^k{}_p) + \hat{\Gamma}_p{}^i{}_k \hat{\Gamma}_q{}^k{}_j - \hat{\Gamma}_q{}^i{}_k \hat{\Gamma}_p{}^k{}_j$$
$$= 2\, \{g^i{}_{[p}\, \hat{L}_{q]j} - g^i{}_j\, \hat{L}_{[pq]} - \ g_{j[p}\, \hat{L}_{q]}{}^i\} + \Theta\, d^i{}_{jpq},$$

$$\hat{\nabla}_p\, \hat{L}_{qj} - \hat{\nabla}_q\, \hat{L}_{pj} = d_i\, d^i{}_{jpq},$$

$$\nabla_i d^i{}_{jkl} = 0.$$

In the Bianchi equation the connection $\nabla = \nabla_g$ **is used, it holds**
$\hat{\Gamma}_i{}^j{}_k = \Gamma_i{}^j{}_k + \delta^j{}_i\, f_k + \delta^j{}_k\, f_i - \eta_{ik}\, \eta^{jl}\, f_l$ **with** $f_i = \frac{1}{n}\, \hat{\Gamma}_i{}^k{}_k$**.**

New gauge freedom. No equations given for Θ **and** d_i**.**

Conformal geodesics.

Conformal geodesic:
curve $x(\tau)$ with 1-form $b(\tau)$ along $x(\tau)$ such that

$$(*_1)\quad (\tilde{\nabla}_{\dot{x}}\dot{x})^\mu + S(b)_\lambda{}^\mu{}_\rho\,\dot{x}^\lambda\,\dot{x}^\rho = 0,$$

$$(*_2)\quad (\tilde{\nabla}_{\dot{x}}b)_\nu - \tfrac{1}{2}\,b_\mu\,S(b)_\lambda{}^\mu{}_\nu\,\dot{x}^\lambda = \tilde{L}_{\lambda\nu}\,\dot{x}^\lambda,$$

where $\tilde{L}_{\mu\nu} = \frac{1}{2}\left(\tilde{R}_{\mu\nu} - \frac{1}{6}\,\tilde{R}\,\tilde{g}_{\mu\nu}\right)$,

$x(\tau)$ **is a conformal invariant: it is still solution after** $\tilde{g} \to \Omega^2\,\tilde{g}$, $\tilde{\nabla} \to \hat{\nabla}$,

Conformal geodesic specified by data $x(\tau_*) = x_*,\ \dot{x}(\tau_*) = \dot{x}_*,\ b(\tau_*) = b_*$.

sign of $\tilde{g}(\dot{x},\dot{x})$ **preserved along** $x(\tau)$**:** $\tilde{\nabla}_{\dot{x}}(\tilde{g}(\dot{x},\dot{x})) = -2 < b, \dot{x} > \tilde{g}(\dot{x},\dot{x})$,

$x(\tau)$, $b(\tau)$ **conf. geod. then** $\bar{x}(\bar{\tau}) = x(\tau(\bar{\tau}))$, $\bar{b}(\bar{\tau})$ **conf. geod, iff**

$\tau = \tau_* + \frac{\bar{\tau}-\bar{\tau}_*}{e+c\,(\bar{\tau}-\bar{\tau}_*)}$, $\quad \bar{b} = b + \frac{1}{\tilde{g}(\dot{x},\dot{x})}\frac{2\,c}{1-c\,(\tau-\tau_*)}\,\dot{x}_\flat$, $\quad e, c, \tau_*, \bar{\tau}_* \in I\!R$, $\quad e \neq 0$.

$(*_3)\quad (\tilde{\nabla}_{\dot{x}}e_k)^\mu + S(b)_\lambda{}^\mu{}_\rho\,\dot{x}^\lambda\,e^\rho{}_k = 0$ **respects** $\tilde{g}(e_i, e_k) = \Theta^2\,\eta_{ik}$ **with**

$(*_4)\quad \tilde{\nabla}_{\dot{x}}\,\Theta = \Theta < b, \dot{x} >, \quad \Theta_* > 0.$

Conformal Gauss gauge (including x^μ, e_k, Θ) based on space-like hypersurface S:

choose coordinates x^α **on** S, $\alpha = 1, 2, 3$,
prescribe smooth data $x_*,\ \dot{x}_*,\ b_*,\ \Theta_* > 0,\ e_{k*}$ **on** S **with:**

$$\dot{x}_* = e_{0*} \perp S,\quad \tilde{g}(e_i, e_k)_* = \Theta_*^2\,\eta_{ik},$$

solve $(*_1)$, $(*_2)$, $(*_3)$, $(*_4)$ **and** $\tilde{\nabla}_{\dot{x}}\,x^\alpha = 0$,
set $x^0 = \tau$, $S = \{\tau = 0\}$.

The slices $\{\tau = const. > 0\}$ are not necessarily orthogonal to $\dot{x} = e_0$!

The data on S are given 'by hand', the rest, in particular the conformal scaling, is determined by the conformal structure.

Conformal geodesics on vacuum solutions.

If Einstein's equations $Ric[\tilde{g}] = \lambda\,\tilde{g}$ hold and $\tilde{g}(\dot{x},\dot{x}) > 0$, then

Scaling and parametrization:

Θ and $d_k = \Theta\, b_\mu\, e^\mu{}_k$ can be integrated along the conformal geodesics. The explicit expressions for Θ and d_k are

$$\Theta = \Theta_* \left(1 + \tau \;< b_*, \dot{x}_* > + \frac{\tau^2}{2}\left(\Theta_*^{-2}\,\frac{\lambda}{6} + \frac{1}{2}\,g^\sharp(b_*, b_*)\right)\right),$$

$$d_0 = \dot{\Theta}, \quad d_a = < b_*, \Theta_*\, e_{a*} >, \;\; a = 1, 2, 3,$$

These expressions encode information on the field equations, in particular (on a smooth, nondegenerate congruence):

$$\nabla_k\,\Theta\,\nabla^k\,\Theta = -\frac{1}{3}\,\lambda \quad \textbf{where} \quad \Theta = 0.$$

For suitable data the expression for Θ admits 2 zeros.

Point sets:

Reparametrize $x(\tau)$ such that $\bar{x}(\bar{\tau}) = x(\tau(\bar{\tau}))$ satisfies $\tilde{g}(\bar{x}', \bar{x}') = 1$.
Write $b = \hat{b} + \zeta\,\dot{x}^\flat$ such that $< \hat{b}, \dot{x} >= 0$. It follows:

on vacuum fields the conformal geodesic equations imply the 'vacuum-adapted conformal geodesic equations'

$$\tilde{\nabla}_{\bar{x}'}\,\bar{x}' = \hat{b}^\sharp, \quad \tilde{\nabla}_{\bar{x}'}\,\hat{b} = \beta^2\,\bar{x}'^\flat,$$

where $\beta^2 \equiv -\tilde{g}^\sharp(\hat{b}, \hat{b}) = (\delta^{ab}\,d_a\,d_b)_* =$ const. along $\bar{x}(\bar{\tau})$,

Thus:

if $\hat{b}$ vanishes at a point, the curve is a $\tilde{g}$-geodesic,

the choice of ζ_* determines the parameterization (slicing),

after a reparameterization conformal geodesics satisfy an equation of third order. Here $\tilde{\nabla}^2_{\bar{x}'}\,\bar{x}' = \beta^2\,\bar{x}$.

The general conformal field equations II.

We call the 'conformal equations based on Weyl connections' in a conformal Gauss gauge the 'general conformal field equations'.

With the 'vacuum expressions for Θ and d_k' (or with the equations governing Θ and d_k) the equations represent a complete system for

$$u = (e^{\mu}{}_{k},\ \hat{\Gamma}_{i}{}^{j}{}_{k},\ \hat{L}_{jk},\ d^{i}{}_{jkl}).$$

A conformal Gauss gauge implies the explicit conditions

$$\dot{x} = e_0 = \partial_\tau, \quad \hat{\Gamma}_{0}{}^{j}{}_{k} = 0, \quad \hat{L}_{0k} = 0.$$

The general conformal field equations thus imply evolution equations of the form

$$\partial_\tau\, e^{\mu}{}_{k} = -\hat{\Gamma}_{k}{}^{l}{}_{0}\, e^{\mu}{}_{l},$$

$$\partial_\tau\, \hat{\Gamma}_{l}{}^{i}{}_{j} = -\hat{\Gamma}_{k}{}^{i}{}_{j}\, \hat{\Gamma}_{l}{}^{k}{}_{0} + g^{i}{}_{0}\, \hat{L}_{lj} - \ g_{j0}\, \hat{L}_{l}{}^{i} + g^{i}{}_{j}\, \hat{L}_{l0} + \Theta\, d^{i}{}_{j0l},$$

$$\partial_\tau\, \hat{L}_{kj} + \hat{\Gamma}_{k}{}^{i}{}_{0}\, \hat{L}_{ij} = d_i\, d^{i}{}_{j0k},$$

$$\nabla_i d^{i}{}_{jkl} = 0.$$

This system implies symmetric hyperbolic 'reduced equations' which preserve the constraints in domains where $\Theta > 0$.

In the conformal Gauss gauge the location of the set $\{\Theta = 0\}$ can be prescribed explictly in terms of the data.

The Penrose proposal, problems and results

Gravitational radiation ?

Einstein (1915) and textbooks discuss "gravitational radiation" in terms of the linearized Einstein equations.

Pirani, Trautman, Sachs, Bondi, Newman, Penrose , ... (ca. 1957 - 1962) : Does there exists a concept of radiation based on the non-linear theory ?

Maxwell theory: Perturbations travel inside or on the null cone. Need to go far out for perturbations to develop into 'waves'. Suggests:

Analyse the propagation of gravitational fields on outgoing null hypersurfaces. Follow the propagation far out.

Where is 'far out', what does it look like, how do we identify 'radiation' if there is no background space ? How do we analyse these questions ?

Specialize: Consider systems of stars, generating the perturbation, which are "far away" from other star systems ("isolated system").

Idealize: Assume that the field approaches in some sense the Minkowski field as the affine parameter $r \to \infty$ along the outgoing null geodesics.

Hope: Higher order quantities have a limit which can be interpreted as representing 'gravitational radiation'.

Gravitational Radiation !

GUESS: $\Psi_0 = O(r^{-5})$ **in a uniform way as** $r \to \infty$.

Formal expansion (in $\frac{1}{r}$**) type analysis gives:**

$\Psi_k = O(r^{k-5})$ **as** $r \to \infty$ **("Sachs peeling").**
$\lim_{r\to\infty} r\,\Psi_4$ **represents the "radiation field".**
The 'Bondi-mass' satisfies a 'mass-loss formula' (and is positive). 'Gravitational waves carry only positive energy'.

In these considerations the null cone structure and associated structures (null hypersurfaces, null geodesics etc.) played a critical role. As an object in itself the null cone structure is arkward to handle. However, in terms of the equivalent 'conformal structure', it can be conveniently analysed.

The critical role of the null cone structure in the studies above was highlighted by the following observation:

Penrose (1963): "The asymptotic behaviour postulated resp. derived above can be characterized geometrically solely in terms of the asymptotic behaviour of the conformal structure".

The more precise statement/proposal following this observation raised some controversy.

The following discussion will largely be concerned with the nature of the proposal, its problems, and the attempt to provide a firm foundation for it.

But it will also show that the consequent abstract analysis of the proposal has practical consequences.

Conformal extensions I.

Construct 'conformal extensions' for the simply connected, conformally flat standard solutions to $Ric[\tilde{g}] = \lambda\, \tilde{g}$ by suitable embeddings.

Target in all 3 cases: the 'Einstein cosmos' $(\bar{M}, g)$ with
$\bar{M} = I\!R \times S^3$**,** $g = d\,s^2 - d\,\omega^2 = d\,s^2 - (d\,\chi^2 + \sin^2\chi\, d\,\sigma^2)$**.**
(Standard line elements: $d\,\omega^2$ **on** S^3**,** $d\,\sigma^2$ **on** S^2**)**

$\lambda < 0$**, de Sitter space:** $\tilde{M} = I\!R \times S^3$**,** $\tilde{g} = dt^2 - \cosh^2 t\, d\,\omega^2$**.**

The map $\tilde{M} \ni (t,\ \vartheta) \xrightarrow{\Phi} (s = \arctan e^t - \frac{\pi}{4},\ \vartheta) \in \tilde{M}' \subset \bar{M}$ **defines a diffeomorphism onto** $\Phi(\tilde{M}) = \tilde{M}' =]-\pi/4, \pi/4[\times S^3$**, in fact a conformal embedding: holds** $\Omega^2\, \Phi^{-1*}\tilde{g} = g$**,** $\Omega = \cos(2\,s) > 0$ **on** $\tilde{M}'$**.**

Observations:

$\Phi(\tilde{M})$ **has a** C^∞ **boundary** $\mathcal{J} \equiv \partial\,(\Phi(\tilde{M})) = \mathcal{J}^+ \cup \mathcal{J}^-$ **in** M**,**

Ω **and** $g = \Omega^2\, \Phi^{-1*}\tilde{g}$ **extend smoothly to** $M = \tilde{M}' \cup \mathcal{J}$**,**

$\Omega = 0,\ d\,\Omega \neq 0$ **on** $\mathcal{J}^\pm = \{s = \pm\pi/4\} \sim S^3$**.**

The 'physical' space-time is finite, the 'conformal boundary at null (and time-like) infinity' at a finite location with respect to the 'conformal metric' g**.**

Convenient analysis of asymptotic behaviour of Maxwell fields.

Convenient analysis of global causal space-time structure, observe e.g. that $\mathcal{J}$ **is space-like.**

Conformal extensions II.

$\lambda > 0$, **anti-de Sitter covering space:**
$\tilde{M} = I\!R^4$, $\tilde{g} = \cosh^2 r\, dt^2 - (dr^2 + \sinh^2 r\, d\sigma^2))$.

The map $\tilde{M} \ni (t, r, \vartheta) \stackrel{\Phi}{\rightarrow} (s = t, \chi = 2 \,\arctan(e^r) - \frac{\pi}{2}, \vartheta) \in \tilde{M}' \subset \bar{M}$,
defines a diffeomorphism onto $\Phi(\tilde{M}) = \tilde{M}' = \{\chi < \pi/2\}$, **in fact a conformal embedding: holds** $\Omega^2\, \Phi^{-1*}\tilde{g} = g$, $\Omega = \cos\chi > 0$ **on** $\tilde{M}'$

Observations:

$\Phi(\tilde{M})$ **has a** C^∞ **boundary** $\mathcal{J}$ **in** M,

Ω **and** $g = \Omega^2\, \Phi^{-1*}\tilde{g}$ **extend smoothly to** $M = \tilde{M}' \cup \mathcal{J}$,

$\Omega = 0,\ d\,\Omega \neq 0$ **on** $\mathcal{J} = \{\chi = \pi/2\} \sim I\!R \times S^2$.

The conformal space-time is finite in space-like directions, the conformal boundary at null (and space-like) infinity is at spatially finite location with respect to the conformal metric g.

The conformal boundary $\mathcal{J}$ **is time-like.**
$\exists$ **non-vanishing solutions of wave equations in AdS which vanish in a neighbourhood of the slice** $\{t = 0\}$.
$\not\exists$ **Cauchy hypersurface in AdS ('not globally hyperbolic').**

Conformal extensions III.

$\lambda = 0$: **Minkowski space** $(\tilde{M} = I\!R^4, \tilde{g} = dt^2 - (dr^2 + r^2\, d\,\sigma^2)$

The map $\Phi : \tilde{M} \rightarrow \tilde{M}' = \{|s \pm \chi| < \pi, \;\; \chi \geq 0\} \subset \bar{M}$ **with inverse**

$$\Phi^{-1}: \qquad t = \frac{\sin s}{\cos s + \cos \chi}, \qquad r = \frac{\sin \chi}{\cos s + \cos \chi},$$

defines a diffeomorphism of $\tilde{M}$ **onto** $\tilde{M}'$**, in fact a conformal embedding:**
$\Omega^2\, \Phi^{-1*}\tilde{g} = g$**,** $\Omega = \cos\, s + \cos\, \chi > 0$ **on** $\tilde{M}'$**.**

Observations:

'Conformal boundary': $\mathcal{J} \equiv \partial\Phi(\tilde{M}) = \mathcal{J}^- \cup \mathcal{J}^+ \cup \{i^-, i^0, i^+\}$ **with:**

'future (past) null infinity' $\mathcal{J}^{\pm} = \{\tau \pm \chi = \pm\pi,\, 0 < \chi < \pi\}$**,**
$\Omega = 0, d\,\Omega \neq 0$ **on** $\mathcal{J}^{\pm} \sim I\!R \times S^2$**, null hypersurfaces,**
'physical' null geodesics acquire endpoints on $\mathcal{J}^{\pm}$**,**

'future (past) time-like infinity' $i^{\pm} = \{\chi = 0, \tau = \pm\pi\}$**,**
endpoint of 'physical' time-like geodesics in the future (past).

'space-like infinity' $i^0 = \{\chi = \pi, \tau = 0\}$**,**
endpoint of 'physical' space-like geodesics.

$\Omega = 0, d\,\Omega = 0, Hess(\Omega) \sim g$ **at** $i^{\pm}$**,** i^0**.**

For detailed investigations near these points sometimes the inversion $x^{\mu} \longrightarrow \frac{x^{\mu}}{x_{\nu}\, x^{\nu}}$ **is useful.**

In all three cases:

the conformal metric g**, the conformal factor** Ω**, and derived fields define solutions to the metric conformal field equations which extend smoothly, as solutions, to the complete Einstein cosmos.**

The Penrose proposal I.

Our observations can be summarized in the generalizing definition

Asymptotic simplicity: *A smooth space-time* $(\tilde{M}, \tilde{g})$ *is called asymptotically simple if there exists a smooth oriented, time-oriented, causal space-time* (M, g) *and on* M *a smooth function* Ω *such that:*

(i) M *is a manifold with boundary* $\mathcal{J}$,
(ii) $\Omega > 0$ *on* $M \setminus \mathcal{J}$ *and* $\Omega = 0$, $d\,\Omega \neq 0$ *on* $\mathcal{J}$,
(iii) there exists an embedding Φ *of* $\tilde{M}$ *onto* $\Phi(\tilde{M}) = M \setminus \mathcal{J}$ *such that*

$$\Omega^2\,\Phi^{-1*}\tilde{g} = g,$$

(iv) each null geodesics of $(\tilde{M}, \tilde{g})$ *acquires two distinct endpoints on* $\mathcal{J}$.

Purely differential geometric restriction on global structure.

Implies that all null geodesics are complete.

Thus $\mathcal{J}$ thus represents *boundary at null infinity*, **generated by ideal endpoints of null geodesics.**

(ii) redundently implies that $\mathcal{J}$ is a smooth hypersurface,
(ii) specifies together with (iii) how precisely $\Phi^{-1*}\tilde{g}$ is to be rescaled to obtain a smooth, non-degenerate metric.

The definition was motivated by the following novel idea

Penrose (1963, 1965) : *Far fields of isolated systems behave like asymptotically simple space-times in the sense that they can be smoothly extended to null infinity, as indicated above, after suitable conformal rescalings.*

(vagueness of formulation the lecturer's . . .)

The Penrose proposal II.

The definition should be applied with understanding, it may require modifications: cf. Schwarzschild-Kruskal solution.

We shall in the following not be worried to much with the completeness condition (iv).

The idea brings out the key geometrical structure in previous investigations. It is independent of any distinguished coordinate systems (like 'Bondi coordinates').

It is of practical interest for analysing space-times:

$$complicated\ \ limits \longleftrightarrow local\ \ differential\ \ geometry$$

It is of interest for analysing physical concepts:

$$approximations\ \ in\ \ (\tilde{M}, \tilde{g}) \longleftrightarrow fields\ \ near\ \ \mathcal{J}$$

(Bondi energy momentum, radiation field, ...)

It is of interest for calculating complete space-times numerically by using the conformal field equations

$$artificial\ \ boundaries \longleftrightarrow finite\ \ conformal\ \ spacetime$$

Field equations + asymptotic simplicity have strong implications:
If $\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu}$ near $\mathcal{J}$ then

$$g^{\mu\nu}\, \nabla_\mu \Omega\, \nabla_\nu \Omega = -\frac{1}{3}\,\lambda, \qquad \nabla_\mu \Omega\, C^\mu{}_{\nu\lambda\rho} = 0 \quad \textbf{on} \ \ \mathcal{J}.$$

The sign of the cosmological constant ($\sim$ term of zeroth order) determines the causal nature of the conformal boundary.

If $\lambda \neq 0$, then $C^\mu{}_{\nu\lambda\rho} = 0$ on $\mathcal{J}$.

The Penrose proposal III.

Central and critical: Asymptotic simplicity is in competition with the implications of the quasi-linear, gauge hyperbolic field equations.

Proposal suggests an extremely sharp characterization of the fall-off behaviour of the gravitational fields of isolated systems

No further strengthening possible without implying essential restrictions.

However: is it going too far already ? What are the criteria ?

The question is not whether C^∞ should be replaced by C^k with some large k but whether there exist extensions of class C^k with k large 'enough' such that e.g. the curvature decays to zero at null infinity.

Penrose (1965) obtains the following remarkable result:

- **$(\tilde{M}, \tilde{g})$ is smooth, solves $\tilde{R}_{\mu\nu} = 0$, and admits a conformal extension (M, g, Ω) with M of class C^4, g and Ω of class C^3,**
- **the Weyl spinor Ψ_{abcd} satisfies $\Omega \, \nabla_{ee'} \nabla^a{}_{a'} \Psi_{abcd} \to 0$ at $\mathcal{J}^+$,**
- **the set of null generators of $\mathcal{J}^+$ is diffeomorphic to S^2**

implies $\Psi_{abcd} \to 0$ on $\mathcal{J}^+$.

Thus, if M is of class C^5, g and Ω are of class C^4 the curvature decays to zero at null infinity.

Is the class of solutions to Einstein's equations which admit such extensions 'sufficiently' rich ?

Construction of extensions I.

How to decide whether a solution $(\tilde{M}, \tilde{g})$ is asymptotically simple ?

In general:
NOT by using the Einstein cosmos or the inversion map.

In a systematic discussion the extension (differential structure, null cone field) must be contructed in terms of intrinsic structures of $(\tilde{M}, \tilde{g})$.

Traditional method: **employs null geodesics, null hypersurfaces, ...**

Schwarzschild line element $r > 2\,m$

$$\tilde{g} = \left(1 - \frac{2\,m}{r}\right) d\,t^2 - \left(1 - \frac{2\,m}{r}\right)^{-1} d\,r^2 - r^2\,d\,\sigma^2$$

in retarded null coordinate $w = t - (r + 2\,m\,\log(r - 2\,m))$

$$\tilde{g} = \left(1 - \frac{2\,m}{r}\right) dw^2 + 2\,dw\,d\,r - r^2 d\sigma^2.$$

With $\rho = r^{-1}$

$$\Omega = \rho \;\; \textbf{and} \;\; g = \Omega^2\,\tilde{g} = (1 - 2\,m\,\rho)\,\rho^2\,d\,w^2 - 2\,dw\,d\,\rho - \rho^2 d\sigma^2.$$

have smooth extensions to $\mathcal{J}^+ \equiv \{\rho = 0\}$.

Alternative method:

de Sitter space: suitable conformal Gauss gauge based on $\tilde{S} = \{t = 0\}$ gives coordinate transformation $\tau = 2\,\tanh\left(\frac{t}{2}\right)$ and conformal factor $\Theta = 1 - \frac{\tau^2}{4}$.

$$\Theta \;\; \textbf{and} \;\; g = \Theta^2\,\tilde{g} = d\,\tau^2 - (1 + \frac{\tau^2}{4})^2\,d\,\omega^2, \quad \textbf{given on} \;\;]-2, 2[,$$

extend smoothly to the sets $\mathcal{J}^\pm \equiv \{\tau = \pm 2\}$.

Construction of extensions II.

Minkowski space:
suitable conformal Gauss gauge based on $\tilde{S} = \{t = 0\}$ gives a coordinate transformation

$$t = \frac{\frac{\tau}{2}}{\cos^2 \frac{\chi}{2} - (\frac{\tau}{2} \sin \frac{\chi}{2})^2}, \quad r = \frac{\sin^2 \frac{\chi}{2} \, (1 + (\frac{\tau}{2})^2)}{\cos^2 \frac{\chi}{2} - (\frac{\tau}{2} \sin \frac{\chi}{2})^2},$$

and a conformal factor

$$\Theta = 2 \, (\cos^2 \frac{\chi}{2} - (\frac{\tau}{2} \sin \frac{\chi}{2})^2)$$

on the manifold

$$\tilde{M} = \{0 < \chi < \pi, \quad \vartheta \in S^2, \quad \tau = \pm \sqrt{\frac{1 + \cos \chi}{1 - \cos \chi}}\}.$$

The fields

$$\Theta \;\; \textbf{and} \;\; g = \Theta^2 \, \tilde{g} = \Theta^2 \, (dt^2 - dr^2 - r^2 \, d\sigma^2) = \omega^2 \, (\frac{1}{\omega^2} \, d\tau^2 - d\chi^2 - \sin^2 \chi \, d\sigma^2),$$

with $\omega = 1 + (\frac{\tau}{2})^2$, extend smoothly to

$$\mathcal{J}^{\pm} = \{0 < \chi < \pi, \quad \vartheta \in S^2, \quad \tau = \pm \sqrt{\frac{1 + \cos \chi}{1 - \cos \chi}}\}.$$

This extension, for which the 1-form b satisfies $< b, \dot{x} > = 0$ on $\{t = 0\}$, does not cover $i^{\pm}$ (conformally invariant statement).
Extensions covering either i^+ or i^- can be constructed by replacing the initial data b by $b + \alpha \, \dot{x}$ with a suitable function α on $\{t = 0\}$.

What happens in the presence of 'strong' curvature ?
Will the curvature necessarily induce the congruence of conformal geodesics to develop caustics ?
Congruences of conformal geodesics can develop caustics which are more severe than those developed by congruences of metric geodesics.

Construction of solutions including their extensions

'There exist smooth (analytic) conformal Gauss systems which cover the complete Schwarzschild-Kruskal space-time and which provide smooth (analytic) conformal extensions to $\mathcal{J}^{\pm}_{1,2}$'.

The underlying congruence of conformal geodesics can be calculated in terms of elliptic integrals.

The regularity of the congruence is shown by deriving and analysing a conformal analogue of the Jacobi equation, which reads in terms of the vacuum-adapted conformal geodesic equations

$$\tilde{\nabla}_X \tilde{\nabla}_X Z = C\,(X,\, Z)\, X + \hat{B}^{\sharp},$$

$$\tilde{\nabla}_X \hat{B} = -\hat{b}\, C\,(X,\, Z) + \alpha\, \tilde{\nabla}_X Z^{\flat} + \gamma\, X^{\flat},$$

here $\alpha, \gamma = const.$ are along the curves and the tangent vector, the deviation vector field, and the deviation 1-form are denoted by

$$X = \partial_{\bar{\tau}}\, \bar{x}, \quad Z = \partial_{\lambda}\, \bar{x}, \quad \hat{B} = \tilde{\nabla}_Z\, \hat{b}.$$

These observations above suggest:

a systematic way of constructing extensions,

a systematic method of constructing solutions including their extensions: In the three cases above the resulting conformal space-times represent smooth solutions to the general conformal field equations.

Even in the Schwarzschild-Kruskal case the reduced equations reduce to systems or ODE's. On a Schwarzschild part $r > 2\,m$ the data can be arranged such that the conformal Gauss systems (including their extensions beyond $\mathcal{J}^{+}$) approach a conformal Gauss system on the (conformally extended) Minkowski space as $m \to 0$.

First issue: existence 'locally near $\mathcal{J}$'.

$\tilde{R}_{\mu\nu} = 0$ **(null infinity light-like):**

'Asymptoytic characteristic initial value problem': data hypersurfaces $\mathcal{N}$, $\mathcal{J}^+_*$ with $\mathcal{N}$ $\sim$ outgoing null hypersurface, $\Sigma = \mathcal{N} \cap \mathcal{J}^+$ space-like 2-surface, $\mathcal{J}^+_*$ $\sim$ part of $\mathcal{J}^+$ in the past of Σ ('the original problem'):

J. Kannar, Proc. Roy Soc. 1996:

For smooth data $\exists_1$ local solution near Σ, i.e. all solutions for which $d^i{}_{jkl}$ has a smooth limit on $\mathcal{J}^+_*$ (and on $\mathcal{N}$) can be characterized.

The freedom to prescribe *null data* on $\mathcal{N}$ and $\mathcal{J}^+_*$ the same as in the characteristic initial value problem for Einstein's equations.

The null data on $\mathcal{J}^+$: *outgoing radiation field*.

$\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu}$**, de Sitter-type case $\lambda < 0$, compact time slices (null infinity space-like):**

For smooth data $\exists_1$ local solution near $\mathcal{J}^-$, i.e. all solutions for which $d^i{}_{jkl}$ has a smooth limit on $\mathcal{J}^-$ can be characterized.

With the exception of the mean intrinsic curvature the freedom to specify data on $\mathcal{J}^-$ the same as in a standard Cauchy problem.

Peculiar feature: the Hamiltonian contraint becomes trivial on $\mathcal{J}^-$.

$\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu}$**, anti-de Sitter-type case $\lambda > 0$, (null infinity time-like):**

All solutions for which $d^i{}_{jkl}$ has a smooth limit on $\mathcal{J}$ can (locally) be characterized in terms of an initial boundary value problem with (standard) initial data on a space-like slice S and boundary data (3-dim Lorentzian conformal structure) on $\mathcal{J}$: $\exists_1$ local solution near S containing a neighbourhood of $S \cap \mathcal{J}$.

Problem more natural than the standard initial boundary value problem for Einstein's vacuum field equations (cf. H.F., G. Nagy, CMP, 1999).

$\exists$ smooth physical solutions with non-smooth boundary data ??

Second issue: Smooth evolution into $\mathcal{J}^+$.

Properties of $d^i{}_{jkl}$ should be deduced, not postulated. We need to control the behaviour of the solutions as they evolve towards null infinity.

Model case:
$\tilde{R}_{\mu\nu} = \lambda\,\tilde{g}_{\mu\nu}$, **de Sitter-type case** $\lambda < 0$, **time slices** $\sim S^3$**:**

'The asymptotic structure of de Sitter space is non-linearly stable under sufficiently small finite perturbations'.

In our conformal Gauss gauge de Sitter space extends as a solution to the general conformal field equations smoothly into the range $-a \leq \tau \leq a$ **for given** $a > 2$**. The conformal factor becomes negative on the boundary of that domain.**

Use the stability properties of the symmetric hyperbolic reduced equations implied by the general conformal field equations to control the perturbed solutions.

In this case null infinity admits a smooth conformal structure and a smooth limit of $d^i{}_{jkl}$ **as a consequence of the evolution process and the assumptions on the initial data.**

An analogous stability result in the Minkowski-type case is not available. In fact, a result of this simplicity cannot be obtained.

Results on the asymptotic conformal structure of asymptotically flat vacuum solutions with vanishing cosmological constant.

Two types of initial value problems.

Most important:

standard Cauchy problem, data on a space-like hypersurface S with asymptotically euclidean end(s).

Standard example:

$\tilde{S} = \{t = 0\} \sim I\!R^3$ **in Minkowski space, induced data** $-\tilde{h} = \tilde{e}$, $\tilde{\chi} = 0$.
By embedding of Minkowski space into the Einstein cosmos:

$$\tilde{S} \to S = \tilde{S} \cup \{i\} \sim S^3, \quad \tilde{e} \to d\omega^2, \quad \tilde{\chi} \to \chi = 0,$$

with i representing space-like infinity for $(\tilde{S}, \tilde{e})$, $d\omega^2 = \Omega^2\, \tilde{e}$ on $\tilde{S}$,
$\Omega \in C^\infty(S)$ with $\Omega = 0$, $d\,\Omega = 0$, $Hess(\Omega) = c\, d\omega^2$ at i, $c \neq 0$.

If $\tilde{h}, \tilde{\chi} \in C^\infty(\tilde{S})$ are asymptotically euclidean vacuum data, then $h = \Omega^2\, \tilde{h}$ is not necessarily in $C^\infty(S)$. But even if $h \in C^\infty(S)$ then

$$d^\mu{}_{\nu\lambda\rho} = O(r^{-3}) \quad \textbf{as} \quad r \to 0 \quad \textbf{unless} \quad m_{ADM} = 0.$$

where r denotes the h-distance from i.

As a preparation study:

hyperboloidal initial value problem, data on a hypersurface H with boundary Σ, thought of as being embedded into an asymptotically simple space-time such that $\Sigma = H \cap \mathcal{J}^+$ and H is space-like. Σ may have several components.

Standard example:

extension of the space-like hyperboloid $\{t^2 - |x|^2 = 1, t > 0\}$ in Minkowski space to $\mathcal{J}^+$ with induced data. Space of constant negative curvature.

Basic differences:

Hyperboloidal problem intrinsically non-time-symmetric,

$$\Omega(p) \sim dist(p, i)^2 \ \textbf{ on } S \quad \textbf{while} \quad \Omega(p) \sim dist(p, \Sigma) \ \textbf{ on } H.$$

Hyperboloidal initial data.

Construction of hyperboloidal data with non-vanishing, constant, mean extrinsic physical curvature following the standard procedure:

L. Andersson, P.T. Chruściel, H. F., CMP, 1992
L. Andersson, P.T. Chruściel, CMP 1994, Diss. Math. 1996.

Prescribe certain 'free data': conformal metric h_{ab} and 'trial $\hat{\chi}_{ab}$' on H,

obtain 'remaining data' (Ω, χ_{ab}) by solving (degenerate) elliptic equations on H,

obtain 'complete conformal data' ($d^i{}_{jkl}, \ldots$) by differentiation and algebra (involves divisions by conformal factor).

Subtleties occur near Σ:

a) for $h_{ab}, \hat{\chi}_{ab} \in C^\infty(H)$ the remaining data and the complete conformal data $\Omega, \chi_{ab}, d^i{}_{jkl}, \ldots \in C^\infty(H \setminus \Sigma)$ have in general a 'polylogarithmic expansions' at Σ, i.e. an expansion in terms of $x^k \log^j x$, where x denotes the h-distance from Σ.
In particular: $d^i{}_{jkl}$ is in general unbounded near Σ.

b) if $h_{ab}, \hat{\chi}_{ab} \in C^\infty(H)$ satisfy (a finite number of) 'regularity conditions' at Σ the remaining data Ω, χ_{ab} as well as the complete conformal data $d^i{}_{jkl}, \ldots$ are smooth on H.

c) if $h_{ab}, \hat{\chi}_{ab} \in C^\infty(H \setminus \Sigma)$ have polylogarithmic expansions at Σ, then the complete conformal data $d^i{}_{jkl}, \ldots$ have polylogarithmic expansions at Σ.

What can be said about the evolution of these data ?

Evolution of hyperboloidal data.

The conformal field equations have been used to obtain the following results:

Case (b):

the data evolve into a solution which admits a smooth 'piece of $\mathcal{J}^+$' in the sense that it satisfies the first three conditions of asymptotic simplicity.

If the data are sufficiently close to Minkowskian hyperboloidal data the solution is null geodesically complete in the future and admits a smooth conformal extension with a regular point i^+ which represents future time-like infinity.

Note: it is a non-trivial consequence of the conformal field equations that they force the null generators of $\mathcal{J}^+$ in a suitable gauge to meet in the future at precisely one point.

Case (a) (Chruściel and Lengard (2001)):

the data evolve into a solution on a manifold $M \sim H \times [0,1[$ with boundary $\partial M \sim \Sigma \times [0,1[$ such that the solution is smooth on $M \setminus \partial M$,

∂M is 'null' in the sense that it is a limit of smooth null hypersurfaces in $M \setminus \partial M$,
$\Omega \to 0$, $d\Omega \to \neq 0$ at ∂M.

The behaviour of the solution near ∂M is controlled in terms of certain weighted Sobolev spaces which admit singularities of the form $d^i{}_{jkl} \sim \frac{1}{x}$ with x the coordinate distance from ∂M.

The solution is expected to admit a polylogarithmic expansion in terms of x.

Which of these solutions can arise by Einstein evolution from asymptotically flat standard Cauchy data ?

Existence of asymptotically simple solutions.

The difficulties at space-like infinity are avoided if the data evolve into a solution which is known explicitly near space-like infinity.

P. Chruściel, E. Delay, CQG, 2002 (cf. also J. Corvino, CMP, 2000):

Given smooth, asymptotically flat, time-symmetric initial data $(h^*_{ab}, I\!R^3)$ **satisfying the vacuum constraint** $R[h^*] = 0$ **and the reflection symmetry** $h^*_{ab}(x) = h^*_{ab}(-x)$**, then for given** $R > 0$, $k \geq k_* > 0$

(i) $\exists\ h_{ab} \in C^k(I\!R^3)$ **such** $h_{ab} = h^*_{ab}$ **for** $|x| < R$, $h_{ab} =$ **Schwarzschild**$_{m \geq 0}$ **for** $|x| > 2\,R$ **and** $R[h] = 0$ **on** $I\!R^3$.

(ii) $\exists\ C^0$**-families** $h_{ab}(\lambda)$, $\lambda \in [0, 1[$, **as above s.t.** $m_h > 0$ **for** $\lambda > 0$, **and** $h_{ab} \to \delta_{ab}$ **as** $\lambda \to 0$ **with fixed** R **and** k.

The time evolution of these data admit smooth hyperboloidal hypersurfaces for which a part coincides with the $|x| \leq 2\,R$ **part of the Cauchy hypersurface and the rest lies in the Schwarzschild part of the evolution. These hyperboloidal hypersurfaces/data can be constructed such that they approach Minkowskian hyperboloidal data as** $m_h \to 0$.

The results on the hyperboloidal initial value problem imply:

There exist non-trivial asymptotically simple solutions which admit complete conformal extension (including regular points $i^{\pm}$**) of class** C^k **for specified** k.

. . . finally, after 40 years . . . . . .

Very special: $d^i{}_{jkl}(i^+) = 0$.

For small m_h**:** $\exists$ **global conformal Gauss systems.**

Possibility to calculate numerically entire space-times by solving the general conformal field equations ?

'$\exists$ **regular** $i^{\pm}$**' not an end in itself. What happens for large** m_h **?**

The problems at space-like infinity.

The standard compactification adds a point i at infinity to the space-like Cauchy hypersurface: $\tilde{S} \equiv \{t = 0\} \to S = \tilde{S} \cup \{i\}$

We have on S (r denoting a radial coordinate with $r(i) = 0$):
a conformal 3-metric $h = O(1)$, which can be chosen to be smooth,
a conformal factor $\Omega = O(r^2)$, which can not be smooth if $m \neq 0$,
a trace free conformal second fundamental form $\chi_{ab} = \Omega^2\,\psi_{ab}$, $\psi_{ab} = O(\frac{1}{r^4})$.

The rescaled conformal Weyl tensor then has the electric part

$$d_{ab} = \Omega^{-2}\left\{D_a\,D_b\Omega - \frac{1}{3}\,h_{ab}\,D^c\,D_c\Omega + \Omega\,s_{ab}\right\}$$

$$-\Omega^3\left\{\psi^c{}_c\,\psi_{ab} - \psi^c{}_a\,\psi_{cb} - \frac{1}{3}\,h_{ab}\,\left((\psi^c{}_c)^2 - \psi^{cd}\,\psi_{cd}\right)\right\} = O(\frac{1}{r^3}),$$

and the magnetic part

$$d^*_{ab} = -2\,D_c\,\Omega\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} - \Omega\,D_c\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} = O(\frac{1}{r^3}).$$

S. Dain, H. F., CMP 2001:
There exists a large class of data with h smooth at i and $\psi_{ab} = O(\frac{1}{r^3})$ such that all data for the conformal field equations have an expansion in r^k at i (i.e. no $\log r$-terms).

If $\psi_{ab} = O(\frac{1}{r^3})$: linear ADM-momentum $= 0$, ADM-angular momentum $\neq 0$ possible. Then $d_{ab} = O(\frac{1}{r^3})$, $d^*_{ab} = O(\frac{1}{r^2})$.
If $\chi_{ab} = 0$ then $d^*_{ab} = 0$ but $d_{ab} = O(\frac{1}{r^3})$, unless $m = 0$.

Consider in the following data on S with h_{ab} smooth and $\psi_{ab} = O(\frac{1}{r^3})$ near i such that all conformal data admit an expansion in r^k.

Nevertheless: the nature of the r-ϑ-relations (ϑ denoting the angular variables) destroys also the smoothness of terms of higher order in an expansion in terms of r^k.
Details of formal expansion type analysis quickly become complicated.

Gauge to be chosen smooth or rough at i ?

Transport of gauge by wave equations generates 'roughness' at $\mathcal{J}^\pm$?

Localization of $\mathcal{J}^\pm = \{\Omega = 0\}$ possible if Ω is 'rough' ?

A new differentiable structure at space-like infinity.

Define a setting which admits a convenient analysis of the r-ϑ-relations.

Choose a conformal scaling for the initial data near i and an oriented h-orthonormal frame $\{e_a\}_{a=1,2,3}$ at i.

Set $e_a(s) = s^c{}_a\, e_c$ with $s = (s^c{}_a) \in SO(3)$, transport $e_a(s)$ parallely along h-geodesics tangent to $e_3(s)$ at i.

Denote by ρ the affine parameter on the geodesics with $< e_3, \rho >= 1$, $\rho(i) = 0$ and denote by $e_a(\rho, s)$ the frame obtained from $e_a(s)$ at value ρ.

Assume $|\rho| < a$, with $B_a(i)$ a convex h-normal neighbourhood. Then:
$]-a, a[\times SO(3) \ni (\rho, s) \stackrel{\Phi}{\to} e_a(\rho, s) \in SO(S)$ defines a smooth embedding into bundle $SO(S)$ of oriented orthonormal frames over S.

The set $\hat{B} \equiv \Phi([0, a[\times SO(3))$ has boundary $I'^0 \equiv \{\rho = 0\} \simeq SO(3)$ and projection $\hat{B} \stackrel{\pi}{\to} B_a(i)$ which maps I'^0 onto i.

The action of $SO(2)$ implies a factorization $\hat{B}' \stackrel{\pi'}{\to} B'' = \hat{B}'/SO(2) \stackrel{\pi''}{\to} B_a(i)$ where π'', which maps $\pi'(I^0) \simeq S^2$ onto i, can be used to identify $B''\setminus\pi'(I^0)$ with $B_a(i) \setminus \{i\}$.

By this identification the inner point i is replaced by a boundary diffeomorphic to the sphere S^2: $B_a(i) \to (B_a(i) \setminus \{i\}) \cup S^2$.

Instead:

Since we will work with a spin frame formalism it is much more natural and convenient to pull back $\hat{B}'$ to the bundle of normalized spin frames, work on 4-dimensional manifold $\hat{B} = [0, a[\times SU(2) \sim [0, a[\times S^3$. Then $\hat{B}$ carries action by $U(1)$, is 'coordinatized' by ρ and $s \in SU(2)$, and has boundary $I^0 = \{0\} \times S^3$ which projects onto $\{i\}$.

A function f on $\hat{B}$ which is invariant under $U(1)$ descends to a function f_* on $B_a(i)$. Since $f \in C^\infty(\hat{B})$ need not imply that $f_* \in C^\infty(B_a(i))$, there are functions on B_a which lift to smooth functions on $\hat{B}$ without being smooth at i.

We shall only consider functions which transform homogeneously under the action of $U(1)$ (have a well defined spin weight).

The regular finite initial value problem near space-like infinity.

To formulate near space-like infinity a useful initial value problem for the general conformal field equations we choose a function $\kappa = \rho\,\mu$ with $\mu \in C^\infty(\hat{B})$ and $\mu = 1$ on I^0 and define a new conformal scaling
$\Omega \to \Theta_* = \kappa^{-1}\,\Omega$, $h \to \kappa^{-2}\,h$, $\ldots$

In this scaling $\Theta = O(\rho)$, the metric coefficients diverge at I^0, but the frame coefficients remain/become smooth.

Further suitable data for the conformal Gauss system give

$$\Theta = \Theta_* \left(1 - (\frac{\tau}{f})^2\right), \qquad d_0 = -2\,\tau\,g, \qquad d_a$$

$$f, g, d_a \in C^\infty(\hat{B}), \quad 1 \leq f < \infty \;\; \textbf{on} \;\; \hat{B}, \;\; f = 1, \;\; g = 0 \;\; \textbf{on} \;\; I^0.$$

The electric and the magnetic parts of the rescaled conformal Weyl tensor transform into

$$d_{ab} = \kappa^3\,\Omega^{-2}\left\{D_a\,D_b\Omega - \frac{1}{3}\,h_{ab}\,D^c\,D_c\Omega + \Omega\,s_{ab}\right\}$$

$$-\kappa^3\,\Omega^3\left\{\psi^c{}_c\,\psi_{ab} - \psi^c{}_a\,\psi_{cb} - \frac{1}{3}\,h_{ab}\,((\psi^c{}_c)^2 - \psi^{cd}\,\psi_{cd})\right\} = O(1),$$

$$d^*_{ab} = -2\,\kappa^3\,D_c\,\Omega\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} - \kappa^3\,\Omega\,D_c\,\psi_{d(a}\,\epsilon_{b)}{}^{cd} = O(1),$$

and lift to smooth functions on $\hat{B}$. In fact all unknowns in the general conformal field equations lift to functions on $\hat{B}$ which can be extended smoothly through I^0 into the region where $\rho \leq 0$.

The hyperbolic equations are to be solved on the 'physical manifold'

$$\tilde{M} = \{\rho > 0, \;\; s \in SU(2), \;\; -f(\rho, s) < \tau < f(\rho, s)\},$$

null infinity is given in our gauge by

$$\mathcal{J}^{\pm} = \{\rho > 0, \;\; s \in SU(2), \;\; \tau = \pm f(\rho, s)\},$$

the *cylinder at space-like infinity* **is given by**

$$I = \{\rho = 0, \;\; s \in SU(2), \;\; |\tau| < 1\}.$$

It touches $\mathcal{J}^{\pm}$ at

$$I^{\pm} = \{\rho = 0, \;\; s \in SU(2), \;\; \tau = \pm 1\}.$$

The evolution near the cylinder at space-like infinity.

In the following we assume that $\chi_{ab} = 0$ near i.

With $w = (e^{\mu}{}_{k}, \hat{\Gamma}_{i}{}^{j}{}_{k}, \hat{L}_{jk})$ and $z = (d^{i}{}_{jkl})$ the reduced equations read

$$(*) \quad A^{\mu}(w)\, \partial_{\mu} z = H(w)\, z, \qquad \partial_{\tau} w = F(w, z),$$

$$A^{0} \textbf{ positive definite on } \hat{B}, \quad {}^{t}\bar{A}^{\mu} = A^{\mu}.$$

They can be extended smoothly, as symmetric hyperbolic system, across I into the domain where $\rho < 0$.

Standard results on solution to symmetric hyperbolic systems then imply the existence of a smooth local solution in a neighbourhood of $\hat{B}$ in $\bar{M} = \{\rho \geq 0, \quad s \in SU(2), \quad -f(\rho, s) < \tau < f(\rho, s)\}$, which covers in particular a neighbbourhood of I^{0} in I.

The cylinder I is thus generated by conformal geometry (as a limit of conformal geodesics) and the field equations.

The details of the reduced equations give

$$A^{\rho} = 0 \quad \textbf{on} \quad I.$$

This implies that the functions

$$u^{p} = \partial_{\rho}^{p}\, u|_{I}, \;\; p = 0, 1, 2, \ldots,$$

satisfy *interior transport equations* **on I and are determined by data on I^{0}.**

Expanding $u^{p} = u^{p}(\tau, s)$ in a suitable function system on $SU(2)$ gives for the expansion coefficients $u^{'p}$ a hierarchy of ODE's along the curves $]-1, 1[\ni \tau \rightarrow (\tau, s) \in I$, which in principle can be solved explicitly.

The operators do not depend on the data, the right hand side of the equation for u^{p+1} depends quadratically on $u^{0}, \ldots, u^{p}$.

The calculation of $A^{\mu}(u)$ on I gives in particular

$$A^{\tau} = diag(1 + \tau, 2, 2, 2, 1 - \tau) \quad \textbf{on} \quad I.$$

The equations thus degenerate on $I^{\pm}$ where $\tau = \pm 1$.

This degeneracy pinpoints the central difficulty of the subject.

A conjecture.

The expansion coefficients u'^p which have been calculated so far have in general terms of the form

$$\left(\frac{1-\tau}{2}\right)^{p-k+2}\left(\frac{1+\tau}{2}\right)^{p+k-2}\left\{e+f\int_0^\tau\frac{d\sigma}{(1-\sigma)^{p-k+3}(1+\sigma)^{p+k-1}}\right\}$$

with constants of integration e, f which follow from the data at I^0. The u^p thus develop logarithmic singularities at I^+. The non-linearity of the equations will also imply terms of the form $(1-\tau)^k\log^j(1-\tau)$.

Inspection of the initial data for u'^p on I^0 shows:

The logarithmic terms observed so far in general do not occur in u'^p for $p \le q_* + 2$ if and only if h satisfies the *regularity condition*

$$\mathcal{S}\left(D_{i_1},\ldots,D_{i_q}B_{jk}(i)\right)=0 \quad \textbf{for} \quad q=0,1,\ldots,q_*,$$

where D denotes the h-connection, $B_{jk} = 1/2\,\epsilon_j{}^{il}\,D_i\,L_{lk}$ the h-Cotton tensor, and $\mathcal{S}$ means: 'take the symmetric trace-free part of ...'.

The conditions are conformally invariant.

Static solutions satisfy these conditions for all q_*.

There exists a large class of data on S^3 which satisfy these conditions.

Conjecture: *There exists an integer $k_* \ge 0$ such that for given $k \ge k_*$ the time evolution of an asymptotically euclidean, time-symmetric, conformally smooth initial data set admits a conformal extension to null infinity of class C^k near space-like infinity, if the regularity condition holds for a certain integer $q_* = q_*(k)$.*

If this is true, the results on the hyperbolidal initial value problem will imply the existence of a large class of asymptotically simple solutions with a smooth conformal structure at null infinity.

The subconjecture.

The calculations seem to hint at an unexpected hidden property of Einstein's equations.

Subconjecture: *If the regularity condition holds for a given* $q_* \geq 0$*, the functions* u^p, $p \leq q_* + 2$*, extend smoothly to* $I^{\pm}$.

J. Kánnár, H.F., JMP 2000: verification for $p \leq 3$.
J. Valiente Kroon, 2002: verification for $p \leq 4$ **for data which are conformally flat near** i.

The analysis will provide an insight into the field equations which goes beyond their conformal properties and their hyperbolicity. It will involve their structure at all orders.

The calculations underlying the analysis provide interesting information on the field and physically relevant quantities near space-like infinity. Examples (assuming the correctnes of the conjecture):

J. Kánnár, H.F., JMP 2000: relate the present gauge to the Bondi gauge and derive a formula for the Newman-Penrose constants

$$G_k = \int_{cut(\mathcal{J}^+)} {}_2\bar{Y}_{2,k}\, \Psi^1_0 \sin\vartheta\, d\vartheta\, d\phi, \quad k = -2, \ldots, 2,$$

in terms of the initial data on S **near** i **which is of the general form**
mass × *quadrupole moment - (dipole moment)*^{2}
and generalizes a formula derived by Newman and Penrose for static solutions. This determines $\phi_{abcd}(i^+)$ **in terms of initial data at** I^0 **if the solution admits a regular point** i^+.

J. Valiente Kroon, arXiv:gr-qc/0206050, June 2002: the calculation of u^p **for** $p \leq 5$ **gives a quite unexpected direct relation between the Bondi mass and the Newman-Penrose coefficients. For solutions arising from data which are conformally flat near** i **the Bondi mass has in terms of a Bondi retarded time** u **an expansion of the form**

$$m_B = m_{ADM} + (2^{7/2} \sum_{k=-2}^{2} |G_k|)\, u^{-7} + O(u^{-8}) \;\; \text{as} \;\; u \to -\infty.$$

Spin-2 fields on Minkowski space near space-like and null infinity.

The proof of the subconjecture appears to be a prerequisite for proving the existence of a general class of asymptotically simple solutions.

Choose $\mu = \mu(\rho)$ with $\mu' < 0$ and linearize the setting and the equations of the regular finite initial value problem at Minkowski space.
The Bianchi part of the reduced equations then implies for the components ϕ_k of rescaled conformal Weyl spinor the equations

$$(1 + (\mu + \rho\,\mu')\,\tau)\,\partial_\tau\,\phi_0 - \rho\,\mu\,\partial_\rho\,\phi_0 + \mu\,X_+\,\phi_1 = -(2\,\mu + 3\,\rho\,\mu')\,\phi_0,$$

$$2\,\partial_\tau\,\phi_1 + \mu\,X_+\,\phi_2 + \mu\,X_-\,\phi_0 = -2\,\mu\,\phi_1,$$

$$2\,\partial_\tau\,\phi_2 + \mu\,X_+\,\phi_3 + \mu\,X_-\,\phi_1 = 0,$$

$$2\,\partial_\tau\,\phi_3 + \mu\,X_+\,\phi_4 + \mu\,X_-\,\phi_2 = 2\,\mu\,\phi_3,$$

$$(1 - (\mu + \rho\,\mu')\,\tau)\,\partial_\tau\,\phi_4 + \rho\,\mu\,\partial_\rho\,\phi_4 + \mu\,X_-\,\phi_3 = (2\,\mu + 3\,\rho\,\mu')\,\phi_4,$$

with $X_\pm = -(Z_2 \pm i\,Z_1)$, where Z_i, $i = 1, 2, 3$, are (real) left invariant vector fields on $SU(2)$ such that $[Z_i, Z_j] = \epsilon_{ijk}\,Z_k$.

This is a symmetric hyperbolic system on the Minkowski space (part)

$$\tilde{M} = \{0 < \rho,\;\; s \in SU(2),\;\; |\tau| < 1/\mu\}$$

and extends as such through $\mathcal{J}^\pm = \{0 < \rho,\; s \in SU(2),\; \tau = \pm 1/\mu\}$ and $I = \{0 = \rho,\; s \in SU(2),\; |\tau| < 1/\mu\}$.

For data which are smooth on $S = \{0 \le \rho,\;\; s \in SU(2),\;\; \tau = 0\}$ the standard energy estimates allow us to show the existence of smooth solutions $\bar{M} = \{0 \le \rho,\;\; s \in SU(2),\;\; |\tau| < 1/\mu\} = \tilde{M} \cup I$.
They do not supply useful information on ϕ_k near $\mathcal{J}^\pm$ because of the degeneracy of the equations on $I^\pm = \{\rho = 0, s \in SU(2), \tau = \pm 1\}$.

A useful estimate I.

Assume that ϕ_k is smooth on $\tilde{M} \cup I$ and solves there the extended spin-2 equations. In the gauge where $\mu \equiv 1$ we have

$$M = \{0 < \rho,\ \ |\tau| < 1\}, \quad \mathcal{J}^{\pm} = \{0 < \rho,\ \ \tau = \pm 1\},$$

and the full (overdetermined) set of spin-2 equations reads

$$0 = A_k \equiv (1+\tau)\,\partial_\tau\,\phi_k - \rho\,\partial_\rho\,\phi_k + X_+\,\phi_{k+1} + (2-k)\,\phi_k,$$

$$0 = B_k \equiv (1-\tau)\,\partial_\tau\,\phi_{k+1} + \rho\,\partial_\rho\,\phi_{k+1} + X_-\,\phi_k + (1-k)\,\phi_{k+1},$$
$$k = 0, \ldots, 3.$$

To define a useful Sobolev norm we consider on $SU(2)$ for a given multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the operator Z^α which is obtained from $Z_1^{\alpha_1}\,Z_2^{\alpha_2}\,Z_3^{\alpha_3}$ by symmetrization and normalization with the factor $(\alpha_1!\,\alpha_2!\,\alpha_3!)^{1/2}\,|\alpha|^{-1/2}$. With $D^{q,p,\alpha} = \partial_\tau^q\,\partial_\rho^p\,Z^\alpha$, the equation

$$\overline{D^{q,p,\alpha}\phi_k}\,D^{q,p,\alpha}\,A_k + D^{q,p,\alpha}\phi_k\,\overline{D^{q,p,\alpha}\,A_k} + \overline{D^{q,p,\alpha}\phi_{k+1}}\,D^{q,p,\alpha}\,B_k + D^{q,p,\alpha}\phi_{k+1}\,\overline{D^{q,p,\alpha}\,B_k} = 0,$$

can be written

$$\partial_\tau\left((1+\tau)\,|D^{q,p,\alpha}\phi_k|^2 + (1-\tau)\,|D^{q,p,\alpha}\phi_{k+1}|^2\right)$$

$$-\partial_\rho\left(\rho\,|D^{q,p,\alpha}\phi_k|^2 - \rho\,|D^{q,p,\alpha}\phi_{k+1}|^2\right)$$

$$+Z^\alpha X_+(\partial_\tau^q\partial_\rho^p\bar{\phi}_k)Z^\alpha(\partial_\tau^q\partial_\rho^p\phi_{k+1}) + Z^\alpha(\partial_\tau^q\partial_\rho^p\bar{\phi}_k)Z^\alpha X_+(\partial_\tau^q\partial_\rho^p\phi_{k+1})$$

$$+Z^\alpha X_-(\partial_\tau^q\partial_\rho^p\phi_k)Z^\alpha(\partial_\tau^q\partial_\rho^p\bar{\phi}_{k+1}) + Z^\alpha(\partial_\tau^q\partial_\rho^p\phi_k)Z^\alpha X_-(\partial_\tau^q\partial_\rho^p\bar{\phi}_{k+1})$$

$$-2\,(p-q+k-2)\,|D^{q,p,\alpha}\phi_k|^2$$

$$+2\,(p-q-k+1)\,|D^{q,p,\alpha}\phi_{k+1}|^2 = 0.$$

The last two terms result from the factors τ and ρ in the differential operators. Their signs depend on the choices of p and q.

A useful estimate II.

Choose $\rho_* > 0$, $0 \le t < 1$, integrate over $N_t = \{0 \le \tau \le t,\ \ 0 \le \rho \le \frac{\rho_*}{1+\tau}\}$ with respect to $d\tau\, d\rho\, d\mu$ with $d\mu$ the normalized Haar measure on $SU(2)$, use Gauss' law, perform summation, use properties of Z^α, to get

$$
\begin{aligned}
&(1+t) \sum_{q'+p'+|\alpha|\le m} \int_{S_t} |D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2 d\rho\, d\mu \\
&+(1-t) \sum_{q'+p'+|\alpha|\le m} \int_{S_t} |D^{q',p',\alpha}(\partial^p_\rho \phi_{k+1})|^2\, d\rho\, d\mu \\
&+2 \sum_{q'+p'+|\alpha|\le m} (p'+p-q'-k+1) \int_{N_t} |D^{q',p',\alpha}(\partial^p_\rho \phi_{k+1})|^2 d\tau\, d\rho\, d\mu \\
&\le \sum_{q'+p'+|\alpha|\le m} \int_{S_0} (|D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2 + |D^{q',p',\alpha}(\partial^p_\rho \phi_{k+1})|^2)\, d\rho\, d\mu \\
&+2 \sum_{q'+p'+|\alpha|\le m} (p'+p-q'+k-2) \int_{N_t} |D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2 d\tau\, d\rho\, d\mu,
\end{aligned}
$$

where $S_t = \{\tau = t,\ \ 0 \le \rho \le \frac{\rho_*}{1+\tau}\}\}$. With $p > m+2$ this implies (i)

$$
\begin{aligned}
&\int_{S_t} (\sum_{q'+p'+|\alpha|\le m} |D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2)\, d\rho\, d\mu \\
&\le \sum_{q'+p'+|\alpha|\le m} \int_{S_0} (|D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2 + |D^{q',p',\alpha}(\partial^p_\rho \phi_{k+1})|^2)\, d\rho\, d\mu \\
&+2\,(p+m+1) \int_{\tau=0}^{t} (\int_{S_\tau} (\sum_{q'+p'+|\alpha|\le m} |D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2)\, d\rho\, d\mu)\, d\tau,
\end{aligned}
$$

whence, by Gronwall's lemma,

$$
\begin{aligned}
&\int_{N_t} (\sum_{q'+p'+|\alpha|\le m} |D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2)\, d\tau\, d\rho\, d\mu, \\
&\le \frac{e^{2\,(p+m+1)\,t} - 1}{2\,(p+m+1)} \left\{ \sum_{q'+p'+|\alpha|\le m} \int_{S_0} (|D^{q',p',\alpha}(\partial^p_\rho \phi_k)|^2 + |D^{q',p',\alpha}(\partial^p_\rho \phi_{k+1})|^2)\, d\rho\, d\mu \right\},
\end{aligned}
$$

for $0 \le k \le 3$, and also (ii)

$$
\begin{aligned}
&2\,(p-m-2) \sum_{q'+p'+|\alpha|\le m} \int_{N_t} |D^{q',p',\alpha}(\partial^p_\rho \phi_4)|^2 d\tau\, d\rho\, d\mu \\
&\le \sum_{q'+p'+|\alpha|\le m} \int_{S_0} (|D^{q',p',\alpha}(\partial^p_\rho \phi_3)|^2 + |D^{q',p',\alpha}(\partial^p_\rho \phi_4)|^2)\, d\rho\, d\mu \\
&+2\,(p+m+1) \sum_{q'+p'+|\alpha|\le m} \int_{N_t} |D^{q',p',\alpha}(\partial^p_\rho \phi_3)|^2 d\tau\, d\rho\, d\mu.
\end{aligned}
$$

Control on behaviour near $\mathcal{J}^+$.

Together these estimates give

$$\int_{N_t}\Big(\sum_{q'+p'+|\alpha|\leq m}|D^{q',p',\alpha}(\partial_\rho^p\phi_k)|^2\Big)\,d\tau\,d\rho\,d\mu \leq$$

$$C\sum_{k=0}^{4}\int_{S_0}\Big(\sum_{q'+p'+|\alpha|\leq m}|D^{q',p',\alpha}(\partial_\rho^p\phi_k)|^2\Big)\,d\rho\,d\mu,$$

$$k=0,\ldots,4, \quad 0\leq t<1,$$

where C depends on p and m but not on $t\in[0,1[$.

Sobolev embedding results imply

$$\partial_\rho^p\,\phi_k\in C^{j,\lambda}(N_1) \quad \textbf{for} \quad p\geq j+6, \quad 0<\lambda\leq 1/2.$$

Writing $J: f\rightarrow J(f)=\int_0^\rho f(\tau,r,s)\,d\,r$ we get by integration

$$\phi_k=\sum_{p=0}^{p-1}\frac{1}{p'\,!}\,\phi_k^{p'}\,\rho^{p'}+J^p(\partial_\rho^p\phi_k) \quad \textbf{on} \quad N_1 \quad \textbf{for} \quad p\geq j+6,$$

resp.

$$\phi_k-\sum_{p'=0}^{p-1}\frac{1}{p'\,!}\,\phi_k^{p'}\,\rho^{p'}\in C^{j,\lambda}(N_1) \quad \textbf{for} \quad p\geq j+6,$$

where the $\phi_k^{p'}(\tau,s)=\partial_\rho^{p'}\,\phi_k|_I$, which are obtained by integrating the transport equations on I, are extended to N_1 as ρ-independent functions.

Thus the overdeterminedness of the equations and the special form of the differential operators, including the existence of the transport equations on I, allow us to get complete control on ϕ_k near $\mathcal{J}^+$.

If the linearized regularity condition is satisfied for all q_*, the $\phi_k^{p'}$ extend smoothly to $I^\pm=\{\rho=0,\tau=\pm1,s\in SU(2)\}$, and $\phi_k\in C^{j,\lambda}(N_1)$ for all j.
Since the ϕ_k^p do in general develop logarithmic singularities, the same can be expected of ϕ_k in the non-linear case.

Can the argument be extended to the non-linear case ?