Einstein equations, conformal structure, and the asymptotic behaviour of space-time

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Null cone structure:
main tool for exploration of gravitational fields in the large.

Differential topology and differential geometry

null cone structure $\leftrightarrow$ *causal structure*

causal sets, boundaries of causal sets, horizons, . . .,

Einstein's field equations $\tilde{R}_{\mu\nu} = \lambda\, \tilde{g}_{\mu\nu} + \kappa\, (T_{\mu\nu} - \frac{1}{2}\, T\, \tilde{g}_{\mu\nu})$

null cone structure $\leftrightarrow$ *characteristics*

positivity conditions, Raychaudhuri equation . . .: 'singularities'
formal expansions, analytical control, numerical calculations, . . .

Differential geometry and field equations

null cone structure $\leftrightarrow$ *conformal structure*

substructure which is invariant under conformal rescalings

$$\tilde{g} \rightarrow g = \Omega^2\, \tilde{g}, \quad \Omega > 0.$$

Ω **small: large 'physical' space-time regions are small w.r.t.** g**-relations.**

If $\Omega \rightarrow 0$ **at 'infinity' the** g**-relations may remain finite on a space-time of infinite 'physical' extent**

Einstein's equations are not conformally covariant ($n = dim\, M$)

$$R_{\nu\rho}[g] = \tilde{R}_{\nu\rho}[\tilde{g}] - \frac{n-2}{\Omega}\, \nabla_\nu\, \nabla_\rho\, \Omega - g_{\nu\rho}\, \left(\frac{1}{\Omega}\, \nabla_\lambda\, \nabla^\lambda\, \Omega - \frac{n-1}{\Omega^2}\, \nabla_\lambda\, \Omega\, \nabla^\lambda\, \Omega\right)$$

principal part retained, right hand side degenerates as $\Omega \rightarrow 0$**.**

Plan

Lecture 1 **Formal properties of the field equations, the conformal field equations.**

Lecture 2 **The Penrose proposal, problems and results.**

Lecture 3 **Results on the asymptotic conformal structure of asymptotically flat vacuum solutions with vanishing cosmological constant.**

Formal properties of the field equations, the conformal field equations.

Conformal geometry:

A conformal rescaling

$$\tilde{g} \to g = \Omega^2 \, \tilde{g}$$

implies transitions $\tilde{\nabla} \to \nabla$ of the Levi-Civita connection and of the Christoffel symbols

$$\tilde{\Gamma}_{\mu}{}^{\rho}{}_{\nu} \to \Gamma_{\mu}{}^{\rho}{}_{\nu} = \tilde{\Gamma}_{\mu}{}^{\rho}{}_{\nu} + S(\Omega^{-1} \, d\,\Omega)_{\mu}{}^{\rho}{}_{\nu}$$

where we write, for any 1-form f,

$$S(f)_{\mu}{}^{\rho}{}_{\nu} \equiv \delta^{\rho}{}_{\mu} \, f_{\nu} + \delta^{\rho}{}_{\nu} \, f_{\mu} - g_{\mu\nu} \, g^{\rho\lambda} \, f_{\lambda}.$$

Decomposition of curvature tensor ($dim \ M = n \geq 4$)

$$R^{\mu}{}_{\nu\lambda\rho} = 2 \, \{ g^{\mu}{}_{[\lambda} \, L_{\rho]\nu} - \, g_{\nu[\lambda} \, L_{\rho]}{}^{\mu} \} + C^{\mu}{}_{\nu\lambda\rho},$$

conformal Weyl tensor $C^{\mu}{}_{\nu\lambda\rho}$, **traces** $L_{\mu\nu} \equiv \dfrac{1}{n-2} \left(R_{\mu\nu} - \dfrac{1}{2\,(n-1)} \, R \, g_{\mu\nu} \right)$.

Bianchi and once contracted Bianchi identity

$$\nabla_{[\delta} \, C^{\mu}{}_{|\nu|\lambda\rho]} = 2 \, (g_{\nu[\lambda} \, \nabla_{\delta} \, L_{\rho]}{}^{\mu} - g^{\mu}{}_{[\lambda} \, \nabla_{\delta} \, L_{\rho]\nu})$$

$$\nabla_{\mu} \, C^{\mu}{}_{\nu\lambda\rho} = (n-3) \, (\nabla_{\lambda} \, L_{\rho\nu} - \nabla_{\rho} \, L_{\lambda\nu})$$

Conformal invariance

$$C^{\mu}{}_{\nu\lambda\rho}[g] = \tilde{C}^{\mu}{}_{\nu\lambda\rho}[\tilde{g}]$$

Conformal non-covariance

$$L_{\mu\nu}[g] = \tilde{L}_{\mu\nu}[\tilde{g}] - \frac{1}{\Omega} \, \nabla_{\mu} \, \nabla_{\nu} \, \Omega + \frac{1}{2\,\Omega^2} \, g_{\mu\nu} \, \nabla_{\rho}\Omega \, \nabla^{\rho}\Omega$$

Conformal covariance

$$\nabla_{\mu} \, (\Omega^{3-n} \, C^{\mu}{}_{\nu\lambda\rho}) = \Omega^{3-n} \, \left\{ \nabla_{\mu} \, C^{\mu}{}_{\nu\lambda\rho} - (n-3) \, \Omega^{n-4} \, \nabla_{\mu} \, \Omega \, (\Omega^{3-n} \, C^{\mu}{}_{\nu\lambda\rho}) \right\}$$

$$= \Omega^{3-n} \, \tilde{\nabla}_{\mu} \, \tilde{C}^{\mu}{}_{\nu\lambda\rho}$$

The metric conformal field equations I.

Assume that $\tilde{g}$ satisfies Einstein's vacuum field equations $\tilde{R}_{\mu\nu} = \lambda\,\tilde{g}_{\mu\nu}$ and thus

$$\tilde{L}_{\mu\nu} = \frac{\lambda}{2\,(n-1)}\,\tilde{g}_{\mu\nu} \quad \textbf{and} \quad \tilde{\nabla}_\mu\,C^\mu{}_{\nu\lambda\rho} = 0.$$

Exploit available conformal covariance to derive a regular system of equations for Ω and the conformal metric $g = \Omega^2\,\tilde{g}$. Set

$$d^\mu{}_{\nu\lambda\rho} \equiv \Omega^{3-n}\,C^\mu{}_{\nu\lambda\rho}$$

The equations above imply for the metric and the conformal curvature fields

$$R^\mu{}_{\nu\lambda\rho} = 2\,\{g^\mu{}_{[\lambda}\,L_{\rho]\nu} - \,g_{\nu[\lambda}\,L_{\rho]}{}^\mu\} + \Omega^{n-3}\,d^\mu{}_{\nu\lambda\rho}$$

$$\nabla_\lambda\,L_{\rho\nu} - \nabla_\rho\,L_{\lambda\nu} = \Omega^{n-4}\,\nabla_\mu\,\Omega\,d^\mu{}_{\nu\lambda\rho}$$

$$\nabla_\mu\,d^\mu{}_{\nu\lambda\rho} = 0 \quad \textbf{(Bianchi equation)}$$

To obtain equations controlling the evolution of the conformal factor use the trace-free part and the trace of the conformal Ricci tensor

$$\nabla_\mu\,\nabla_\nu\,\Omega = -\Omega\,L_{\mu\nu} + s\,g_{\mu\nu},$$

$$\frac{\lambda}{n-1} = 2\,\Omega\,s - \nabla_\rho\,\Omega\,\nabla^\rho\,\Omega.$$

with scalar field

$$s \equiv \frac{1}{n}\,\nabla_\lambda\,\nabla^\lambda\,\Omega + \frac{1}{2\,n\,(n-1)}\,R\,\Omega$$

The equations above imply the integrability condition

$$\nabla_\mu\,s = -\nabla^\nu\,\Omega\,L_{\nu\mu}$$

The metric conformal field equations II.

Unknown

$$u = (g, \Omega, s, L_{\mu\nu}, d^{\mu}{}_{\nu\lambda\rho})$$

Metric conformal field equations

$$R^{\mu}{}_{\nu\lambda\rho} = 2\,\{g^{\mu}{}_{[\lambda}\, L_{\rho]\nu} - \, g_{\nu[\lambda}\, L_{\rho]}{}^{\mu}\} + \Omega^{n-3}\, d^{\mu}{}_{\nu\lambda\rho}$$

$$\nabla_{\mu}\, \nabla_{\nu}\, \Omega = -\Omega\, L_{\mu\nu} + s\, g_{\mu\nu}$$

$$\nabla_{\mu}\, s = -\nabla^{\nu}\, \Omega\, L_{\nu\mu}$$

$$\nabla_{\lambda}\, L_{\rho\nu} - \nabla_{\rho}\, L_{\lambda\nu} = \Omega^{n-4}\, \nabla_{\mu}\, \Omega\, d^{\mu}{}_{\nu\lambda\rho}$$

$$\nabla_{\mu}\, d^{\mu}{}_{\nu\lambda\rho} = 0$$

$$\frac{\lambda}{n-1} = 2\,\Omega\, s - \nabla_{\rho}\,\Omega\,\nabla^{\rho}\,\Omega$$

On a connected set the last equation follows if it holds at one point.

Conformal gauge freedom: equations are invariant under

$$g \longrightarrow g^{*} = \theta^{\frac{4}{n-2}}\, g, \quad \Omega \longrightarrow \Omega^{*} = \theta^{\frac{2}{n-2}}\, \Omega, \quad d^{\mu}{}_{\nu\lambda\rho} \longrightarrow d^{*\mu}{}_{\nu\lambda\rho} = \theta^{-\frac{2}{n-2}}\, d^{\mu}{}_{\nu\lambda\rho},$$

$$s \longrightarrow s^{*} = \theta^{-\frac{2}{n-2}}\, s + \frac{2}{n\,(n-2)}\, \theta^{-\frac{2\,(n-1)}{n-2}}\, (2\,\Omega\, \nabla_{\mu}\theta\, \nabla^{\mu}\theta + n\,\theta\, \nabla_{\mu}\theta\, \nabla^{\mu}\Omega),$$

$$L_{\mu\nu} \longrightarrow L^{*}_{\mu\nu} = L_{\mu\nu}$$

$$-\frac{2}{(n-2)^2}\, \theta^{-2}\, ((n-2)\,\theta\, \nabla_{\mu}\, \nabla_{\nu}\, \theta - n\, \nabla_{\mu}\, \theta\, \nabla_{\nu}\, \theta + g_{\mu\nu}\, \nabla_{\rho}\theta\, \nabla^{\rho}\theta).$$

In particular

$$L_{\mu}{}^{\mu} \longrightarrow L^{*}_{\mu}{}^{\mu} = \theta^{-\frac{n+2}{n-2}} \left(\theta\, L_{\mu}{}^{\mu} - \frac{2}{n-2}\, \nabla_{\mu}\, \nabla^{\mu}\, \theta\right).$$

Suggests to remove the gauge freedom by prescribing the function $R = 2\,(n-1)\, L_{\mu}{}^{\mu}$ **as the conformal gauge source function.**

Then Ω **is determined implicitly by a wave equation.**

The Bianchi equation I.

The vacuum Bianchi identity

$$\tilde{\nabla}_\delta \, C^\mu{}_{\nu\lambda\rho} + \tilde{\nabla}_\rho \, C^\mu{}_{\nu\delta\lambda} + \tilde{\nabla}_\lambda \, C^\mu{}_{\nu\rho\delta} = 0$$

implies in $n \geq 4$ dimensions the system of wave equations

$$\tilde{\nabla}_\delta \, \tilde{\nabla}^\delta \, C^\mu{}_{\nu\tau\rho} = 2 \, \{ C^\mu{}_{\pi\delta\rho} \, C^\pi{}_{\nu\tau}{}^\delta + C^\mu{}_{\pi\tau\delta} \, C^\pi{}_{\nu\rho}{}^\delta + C^\mu{}_{\nu\pi\delta} \, C^\pi{}_{\tau\rho}{}^\delta \} + 2 \, \lambda \, C^\mu{}_{\nu\tau\rho}$$

and is thus always 'intrinsically hyperbolic'.

$n = 4$**: Bianchi identity $\Leftrightarrow$ contracted Bianchi identity**

$$\tilde{\epsilon}_\pi{}^{\delta\tau\rho} \tilde{\nabla}_\delta \, C^\mu{}_{\nu\tau\rho} = 2 \, \tilde{\nabla}_\delta \, C^{*\mu}{}_{\nu\pi}{}^\delta = 2 \, \tilde{\nabla}_\delta \, {}^*C^\mu{}_{\nu\pi}{}^\delta = - \, \tilde{\epsilon}^\mu{}_\nu{}^{\tau\rho} \, \tilde{\nabla}_\delta \, C^\delta{}_{\pi\tau\rho}$$

thus: Bianchi equation $\nabla_\delta \, d^\delta{}_{\pi\tau\rho} = 0$ also 'intrinsically hyperbolic'.

Conformal Weyl tensor: $u = \frac{1}{12} \, n^2 \, (n^2 - 1) - \frac{1}{2} \, n \, (n+1)$ **independent components,**

Contracted Bianchi id.: $e = \frac{1}{3} \, n \, (n-2) \, (n+2)$ **independent equations,**

$c = \frac{1}{2} \, n \, (n-1)$ **constraints**

($\tilde{\nabla}_\delta \, C^\delta{}_{0\tau\rho} = 0$ **in Gauss coordinates)**

$n \geq 5$**:** $e - c < u$ **evolution equations in the contracted Bianchi identity.**

$\nabla_\delta \, d^\delta{}_{\pi\tau\rho} = 0$ **does not provide hyperbolic evolution equations.**

$\exists$ **regular, hyperbolic conformal evolution equations if $n \geq 5$?**

$\exists$ **differences in asymptotic behaviour between $n = 4$ and $n \geq 5$?**

Assume from now on: $n = 4$

The Bianchi equation II.

There are various ways to exhibit the hyperbolicity of the Bianchi equation.

If the function t defines a space-like foliation, we write

$$g = (N\,d\,t)^2 - h_{ij}\,(S^i\,d\,t + d\,x^i)\,(S^j\,d\,t + d\,x^j)$$

$$n^\mu = \frac{1}{N}\,(\delta^\mu{}_0 - S^\mu), \quad a_\mu = \frac{1}{N}\,D_\mu\,N, \quad h^\mu{}_\nu = g^\mu{}_\nu - n^\mu\,n_\nu, \quad \chi_{\mu\nu} = \frac{1}{2}\,\mathcal{L}_n\,h_{\mu\nu}$$

$$C_{\mu\nu\lambda\rho} = 2\,\{l_{\mu[\lambda}\,E_{\rho]\nu} - l_{\nu[\lambda}\,E_{\rho]\mu} - n_{[\lambda}\,B_{\rho]\tau}\,\epsilon^\tau{}_{\mu\nu} - n_{[\mu}\,B_{\nu]\tau}\,\epsilon^\tau{}_{\lambda\rho}\}$$

and denote by D the h Levi-Civita operator on $\{t = const.\}$,

we obtain from the Bianchi equations the constraints

$$D^i\,E_{ij} - B_{ik}\,\chi^i{}_l\,\epsilon^{kl}{}_j = 0$$

$$D^i\,B_{ij} + E_{ik}\,\chi^i{}_l\,\epsilon^{kl}{}_j = 0,$$

which are uniquely determined by the hypersurface $\{t = const.\}$

and propagation equations

$$\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,E_{ij} + D_k\,B_{l(i}\,\epsilon_{j)}{}^{kl} - 3\,E^k{}_{(i}\,\chi_{j)k} + \chi_k{}^k\,E_{ij} - \epsilon_i{}^{kl}\,E_{km}\,\chi_{ln}\,\epsilon_j{}^{mn}$$
$$+2\,a_k\,B_{l(i}\,\epsilon_{j)}{}^{kl} = 0,$$

$$\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,B_{ij} - D_k\,E_{l(i}\,\epsilon_{j)}{}^{kl} - 3\,B^k{}_{(i}\,\chi_{j)k} + \chi_k{}^k\,B_{ij} - \epsilon_i{}^{kl}\,B_{km}\,\chi_{ln}\,\epsilon_j{}^{mn}$$
$$-2\,a_k\,E_{l(i}\,\epsilon_{j)}{}^{kl} = 0,$$

which are not unique.

The latter imply the manifestly symmetric hyperbolic system

$$h^{k(i}h^{j)l}\,\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,E_{ij} + \frac{1}{2}\,(\epsilon^{ij(k}\,h^{l)m} + \epsilon^{im(k}\,h^{l)j})\,D_i\,B_{jm} + ... = 0,$$

$$h^{k(i}h^{j)l}\,\frac{1}{N}\,(\partial_t - \mathcal{L}_S)\,B_{ij} - \frac{1}{2}\,(\epsilon^{ij(k}\,h^{l)m} + \epsilon^{im(k}\,h^{l)j})\,D_i\,E_{jm} + ... = 0.$$

The Bianchi equation III.

In the spin frame formalism $d^{\mu}{}_{\nu\lambda\rho}$ is represented by a symmetric spinor field ϕ_{abcd}

The Bianchi equation reads

$$0 = \Lambda_{abca'} \equiv \epsilon^{ef}\, \nabla_{a'f}\, \phi_{abce}$$

with $a, b, c, \ldots, a', b', c' \ldots = 0, 1$, and $\epsilon^{ab} = \epsilon^{[ab]}$, $\epsilon^{01} = 1$ $\epsilon^{ab}\, \epsilon_{ac} = \delta^{b}{}_{c}$, $\nabla_{aa'}$ the covariant derivative in the direction of $e_{aa'}$, where

$$e_{00'} = 1/\sqrt{2}\,(e_0 + e_3), \;\; e_{11'} = 1/\sqrt{2}\,(e_0 - e_3),$$

$$e_{01'} = 1/\sqrt{2}\,(e_1 + i\, e_2), \;\; e_{10'} = 1/\sqrt{2}\,(e_1 - i\, e_2)$$

with a frame field e_k satisfying $g(e_i, e_k) = \eta_{ik}$ such that $g(e_{aa'}, e_{bb'}) = \epsilon_{ab}\, \epsilon_{a'b'}$.

With $\tau^{aa'} = \delta_0{}^{a}\, \delta_{0'}{}^{a'} + \delta_1{}^{a}\, \delta_{1'}{}^{a'}$, $P = \tau^{aa'}\, \nabla_{aa'} = \sqrt{2}\, \nabla_{e_0}$, $\mathcal{D}_{ab} = \tau_{(a}{}^{a'}\, \nabla_{b)a'} \perp P$:

$$\Lambda_{abca'} = 0 \quad \textbf{iff} \quad \Lambda_{abcd} \equiv \Lambda_{abca'}\, \tau_d{}^{a'} = 0 \;\; \textbf{iff}$$

$$0 = \Lambda_{abf}{}^{f} = \mathcal{D}^{ef}\, \phi_{abef} \quad \textbf{(constraints)}, \quad 0 = \Lambda_{(abcd)} = -\frac{1}{2}\, P\, \phi_{abcd} + \mathcal{D}_{(d}{}^{f}\, \phi_{abc)f}$$

with symmetric hyperbolic version (10 real equations)

$$-\binom{4}{a+b+c+d}\, \Lambda_{(abcd)} = 0.$$

Further symmetric hyperbolic systems:

$$0 = 2\, a\, \Lambda_{(0001')},$$

$$0 = (c - d)\, \Lambda_{(0011')} - 2\, a\, \Lambda_{(0000')},$$

$$0 = (c + d)\, \Lambda_{(0111')} - (c - d)\, \Lambda_{(0010')},$$

$$0 = 2\, e\, \Lambda_{(1111')} - (c + d)\, \Lambda_{(0110')},$$

$$0 = -2\, e\, \Lambda_{(1110')},$$

with $a, c, e > 0$, $-(2\, e + c) < d < 2\, a + c$.

The conformal constraints I.

S space-like hypersurface, $\{e_k\}_{k=0,1,2,3}$ frame near S, $g(e_j, e_k) = \eta_{jk}$, $e_0 \perp S$
Write tensors in this frame, assume $a, b, c, .. = 1, 2, 3$ and sum convention.
Denote by D the Levi-Civita connection of the induced metric h, set

$$h_{ab} = h(e_a, e_b), \quad \epsilon_{abc} = \epsilon_{[abc]}, \quad \textbf{with } \epsilon_{123} = 1$$

$$\Sigma = e_0(\Omega), \quad \chi_{ab} = g(\nabla_{c_a} e_0, c_b), \quad L_a = L_{a0}, \quad d_{ab} = d_{a0b0}, \quad d^*_{ab} = d^*_{a0b0}$$

such that

$$d_{ab} = d_{(ab)}, \quad d_a{}^a = 0, \quad d^*_{ab} = d^*_{(ab)}, \quad d^*_a{}^a = 0, \quad d_{abcd} = 2\{h_{a[c}d_{d]b} + h_{b[d}d_{c]a}\}$$

Interior equations implied by metric conformal field equations on S:

$$^3R_{ab} = \chi_c{}^c \chi_{ab} - \chi_{ca}\,\chi_b{}^c + \Omega\, d_{ab} + L_{ab} + h_{ab}\, L_c{}^c$$

$$D_b \chi_{ca} - D_c \chi_{ba} = \Omega\, d^*_{ae}\, \epsilon^e{}_{bc} + 2\, h_{a[b}\, L_{c]}$$

$$D_a\, D_b \Omega = -\Sigma\, \chi_{ab} - \Omega\, L_{ab} + s\, h_{ab}$$

$$D_a \Sigma = \chi_a{}^c D_c \Omega - \Omega\, L_a$$

$$D_a s = -D^b \Omega\, L_{ba} - \Sigma\, L_a$$

$$D_a L_{bc} - D_b L_{ac} = D^e \Omega\, d_{ecab} - \Sigma\, d^*_{ce}\, \epsilon^e{}_{ab} + 2\, \chi_{c[a}\, L_{b]}$$

$$D_a L_b - D_b L_a = D^f \Omega\, d^*_{fe}\, \epsilon^e{}_{ab} + 2\, \chi_{[a}{}^c L_{b]c}$$

$$D^a d^*_{ab} = -\chi^c{}_e\, d_{cf}\, \epsilon_b{}^{ef}$$

$$D^a d_{ab} = \chi^c{}_e\, d^*_{cf}\, \epsilon_b{}^{ef}$$

$$\lambda = 6\, \Omega\, s - 3\, \Sigma^2 - 3 D_a \Omega\, D^a \Omega$$

Conformal constraints complicated because of integrability conditions and conformal rescaling. The standard vacuum constraints read

$$^3R = (\chi_c{}^c)^2 - \chi_{cd}\, \chi^{cd} + 2\, \lambda, \qquad D_a \chi_c{}^a - D_c \chi_a{}^a = 0.$$

The conformal constraints II.

Standard vacuum constraints

$$^3R = (\chi_c{}^c)^2 - \chi_{cd}\,\chi^{cd} + 2\,\lambda, \qquad D_a\chi_c{}^a - D_c\chi_a{}^a = 0.$$

The conformal field equations include integrability conditions, thus even for $\Omega \equiv 1$ we find the more complicated equations

$$^3R_{ab} = \chi_c{}^c\,\chi_{ab} - \chi_{ca}\,\chi_b{}^c + 2/3\,\lambda\,h_{ab} + d_{ab}$$

$$D_b\chi_{ca} - D_c\chi_{ba} = d^*_{ae}\,\epsilon^e{}_{bc}$$

$$D^a d^*_{ab} = -\chi^c{}_e\,d_{cf}\,\epsilon_b{}^{ef}$$

$$D^a d_{ab} = \chi^c{}_e\,d^*_{cf}\,\epsilon_b{}^{ef}$$

Standard strategy to provide solutions:

Prescribe free data: conformal metric $\hat{h}_{ab}$, 'trial' $\hat{\chi}_{ab}$,

Solve linear resp. semi-linear elliptic equations to obtain solution to the standard constraints (Lichnerowicz, York, Choquet-Bruhat, ...)

Use first subsystem above to define d_{ab} and d^*_{ab}.
The other equations will then be satisfied.

Procedure works for conformal constraints similarly.
Delicate: behaviour near $\Omega = 0$.

Different strategy:

Prescribe free data: 'trial' $\hat{d}_{ab}$, $\hat{d}^*_{ab}$

Read the system as a 3-dimensional Einstein equation with source fields

$$d_{ab} = D_a\,X_b + D_b\,X_a - \frac{2}{3}\,h_{ab}\,D_c\,X^c + \hat{d}_{ab}$$

$$d^*_{ab} = D_a\,X^*_b + D_b\,X^*_a - \frac{2}{3}\,h_{ab}\,D_c\,X^{*c} + \hat{d}^*_{ab}$$

Solve quasi-linear elliptic system for h_{ab}, χ_{ab}, X_a, X^*_a.

First task: formulate well posed elliptic boundary value problems.

A. Butscher (2002): Stability of asymptotically euclidean vacuum solution $h_{ab} = -\delta_{ab}$, $\chi_{ab} = 0$, $d_{ab} = 0$, $d^*_{ab} = 0$.

Gauge conditions, hyperbolicity, and local evolution.

The conformal field equations pose the same problems and offer the same flexibility as Einstein's equations in their standard form.

$\exists$ various ways to handle conformal, coordinate, and frame gauge freedom:
Implicit wave equations for conformal factor, coordinates, frame field, or geometrical gauge with explicit algebraic conditions on unknown, ...

All techniques developed for Einstein's field equations may be tried:
Writing formally $\kappa\, T_{\mu\nu} \equiv 2\,(L_{\mu\nu} - L\, g_{\mu\nu})$, the metric conformal field equations take the form of

'Einstein equations with matter fields'

$$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} = \kappa\, T_{\mu\nu}$$

$$\nabla_\mu \nabla_\nu \Omega = -\frac{\kappa}{2}\,(T_{\mu\nu} - \frac{1}{3}\, T\, g_{\mu\nu})\,\Omega + s\, g_{\mu\nu}$$

$$\nabla_\mu\, s = -\frac{\kappa}{2}\,(T_{\mu\nu} - \frac{1}{3}\, T\, g_{\mu\nu})\,\nabla^\nu\,\Omega$$

$$\kappa\,(\nabla_{[\lambda} T_{\rho]\nu} - \frac{1}{3}\,\nabla_{[\lambda} T\, g_{\rho]\nu}) = \nabla_\mu \Omega\, d^\mu{}_{\nu\lambda\rho}$$

$$\nabla_\mu\, d^\mu{}_{\nu\lambda\rho} = 0$$

(ensure geometrical role of $T_{\mu\nu}$ and $d^\mu{}_{\nu\lambda\rho}$)

$\exists$ hyperbolic systems in the tensor, frame, and spin frame formalism.

$\exists$ various ways to formulate well posed local initial value problems for the metric conformal field equations (preserving constraints and gauge conditions)

Specific for the conformal field equations:

Various global and semi-global geometric problems for Einstein's field equations can be/have been recast as well posed
−characteristics initial value problems,
−Cauchy problems,
−initial boundary value problems
for the conformal field equations, involving finite domains where Ω changes sign.

The general conformal field equations I.

Besides conformal rescalings $\tilde{g} \to g = \Theta^2 \, \tilde{g}$
consider also Weyl connections $\hat{\nabla} = \nabla_g + S(f)$ **with arbitrary 1-form** f**,**
torsion free, respect conformal class $\mathcal{C}_g$**:** $\hat{\nabla}_\rho \, g_{\mu\nu} = -2 \, f_\rho \, g_{\mu\nu}$**.**

Representation of 1-form depends on conformal scaling of metric
$\nabla_g = \nabla_{\tilde{g}} + S(\Theta^{-1} \, \nabla\Theta)$**,** $\hat{\nabla} = \nabla_{\tilde{g}} + S(\Theta^{-1} \, d)$ **with** $d = \Theta \, f + \nabla\Theta$**.**

Write fields in frame $\{e_k\}_{k=0,\ldots,3}$ **with** $g(e_i, e_k) = \eta_{ik}$**. Set**

$$\hat{\nabla}_i e_j \equiv \hat{\nabla}_{e_i} e_j = \hat{\Gamma}_i{}^k{}_j \, e_k, \quad d^i{}_{jkl} = \Theta^{-1} \, C^i{}_{jkl},$$

$$\hat{L}_{jk} = 1/2 \, \hat{R}_{(jk)} - 1/4 \, \hat{R}_{[jk]} - 1/12 \, \hat{R} \, g_{jk}$$

The vacuum field equation $Ric[\tilde{g}] = \lambda \, \tilde{g}$ **then imply for**

$$u = (e^\mu{}_k, \; \hat{\Gamma}_i{}^j{}_k, \; \hat{L}_{jk}, \; d^i{}_{jkl}),$$

the 'conformal equations based on Weyl connections'

$$[e_p, e_q] = (\hat{\Gamma}_p{}^l{}_q - \hat{\Gamma}_q{}^l{}_p) \, e_l,$$

$$e_p(\hat{\Gamma}_q{}^i{}_j) - e_q(\hat{\Gamma}_p{}^i{}_j) - \hat{\Gamma}_k{}^i{}_j(\hat{\Gamma}_p{}^k{}_q - \hat{\Gamma}_q{}^k{}_p) + \hat{\Gamma}_p{}^i{}_k \hat{\Gamma}_q{}^k{}_j - \hat{\Gamma}_q{}^i{}_k \hat{\Gamma}_p{}^k{}_j$$
$$= 2 \, \{g^i{}_{[p} \, \hat{L}_{q]j} - g^i{}_j \, \hat{L}_{[pq]} - \; g_{j[p} \, \hat{L}_{q]}{}^i\} + \Theta \, d^i{}_{jpq},$$

$$\hat{\nabla}_p \, \hat{L}_{qj} - \hat{\nabla}_q \, \hat{L}_{pj} = d_i \, d^i{}_{jpq},$$

$$\nabla_i d^i{}_{jkl} = 0.$$

In the Bianchi equation the connection $\nabla = \nabla_g$ **is used, it holds**
$\hat{\Gamma}_i{}^j{}_k = \Gamma_i{}^j{}_k + \delta^j{}_i \, f_k + \delta^j{}_k \, f_i - \eta_{ik} \, \eta^{jl} \, f_l$ **with** $f_i = \frac{1}{n} \, \hat{\Gamma}_i{}^k{}_k$**.**

New gauge freedom. No equations given for Θ **and** d_i**.**

Conformal geodesics.

Conformal geodesic:
curve $x(\tau)$ with 1-form $b(\tau)$ along $x(\tau)$ such that

$$(*_1) \quad (\tilde{\nabla}_{\dot{x}}\dot{x})^\mu + S(b)_\lambda{}^\mu{}_\rho\,\dot{x}^\lambda\,\dot{x}^\rho = 0,$$

$$(*_2) \quad (\tilde{\nabla}_{\dot{x}}b)_\nu - \tfrac{1}{2}\,b_\mu\,S(b)_\lambda{}^\mu{}_\nu\,\dot{x}^\lambda = \tilde{L}_{\lambda\nu}\,\dot{x}^\lambda,$$

where $\tilde{L}_{\mu\nu} = \frac{1}{2}\left(\tilde{R}_{\mu\nu} - \frac{1}{6}\,\tilde{R}\,\tilde{g}_{\mu\nu}\right)$,

$x(\tau)$ **is a conformal invariant: it is still solution after** $\tilde{g} \to \Omega^2\,\tilde{g}$, $\tilde{\nabla} \to \hat{\nabla}$,

Conformal geodesic specified by data $x(\tau_*) = x_*,\ \dot{x}(\tau_*) = \dot{x}_*,\ b(\tau_*) = b_*$.

sign of $\tilde{g}(\dot{x},\dot{x})$ **preserved along** $x(\tau)$: $\tilde{\nabla}_{\dot{x}}(\tilde{g}(\dot{x},\dot{x})) = -2 < b, \dot{x} > \tilde{g}(\dot{x},\dot{x})$,

$x(\tau)$, $b(\tau)$ **conf. geod. then** $\bar{x}(\bar{\tau}) = x(\tau(\bar{\tau}))$, $\bar{b}(\bar{\tau})$ **conf. geod, iff**

$$\tau = \tau_* + \frac{\bar{\tau}-\bar{\tau}_*}{e+c\,(\bar{\tau}-\bar{\tau}_*)}, \qquad \bar{b} = b + \frac{1}{\tilde{g}(\dot{x},\dot{x})}\frac{2\,c}{1-c\,(\tau-\tau_*)}\ \dot{x}_\flat, \qquad e, c, \tau_*, \bar{\tau}_* \in I\!R, \quad e \neq 0.$$

$(*_3) \quad (\tilde{\nabla}_{\dot{x}}e_k)^\mu + S(b)_\lambda{}^\mu{}_\rho\,\dot{x}^\lambda\,e^\rho{}_k = 0$ **respects** $\tilde{g}(e_i, e_k) = \Theta^2\,\eta_{ik}$ **with**

$(*_4) \quad \tilde{\nabla}_{\dot{x}}\,\Theta = \Theta < b, \dot{x} >, \quad \Theta_* > 0.$

Conformal Gauss gauge (including x^μ, e_k, Θ) based on space-like hypersurface S:

choose coordinates x^α on S, $\alpha = 1, 2, 3$,
prescribe smooth data $x_*,\ \dot{x}_*,\ b_*,\ \Theta_* > 0,\ e_{k*}$ **on S with:**

$$\dot{x}_* = e_{0*} \perp S, \quad \tilde{g}(e_i, e_k)_* = \Theta_*^2\,\eta_{ik},$$

solve $(*_1)$, $(*_2)$, $(*_3)$, $(*_4)$ **and** $\tilde{\nabla}_{\dot{x}}\,x^\alpha = 0$,
set $x^0 = \tau,\ \ S = \{\tau = 0\}$.

The slices $\{\tau = const. > 0\}$ are not necessarily orthogonal to $\dot{x} = e_0$!

The data on S are given 'by hand', the rest, in particular the conformal scaling, is determined by the conformal structure.

Conformal geodesics on vacuum solutions.

If Einstein's equations $Ric[\tilde{g}] = \lambda\, \tilde{g}$ hold and $\tilde{g}(\dot{x}, \dot{x}) > 0$, then

Scaling and parametrization:

Θ and $d_k = \Theta\, b_\mu\, e^\mu{}_k$ can be integrated along the conformal geodesics. The explicit expressions for Θ and d_k are

$$\Theta = \Theta_* \left(1 + \tau\, < b_*, \dot{x}_* > + \frac{\tau^2}{2} \left(\Theta_*^{-2}\, \frac{\lambda}{6} + \frac{1}{2}\, g^\sharp(b_*, b_*)\right)\right),$$

$$d_0 = \dot{\Theta}, \quad d_a = < b_*, \Theta_*\, e_{a*} >, \ \ a = 1, 2, 3,$$

These expressions encode information on the field equations, in particular (on a smooth, nondegenerate congruence):

$$\nabla_k\, \Theta\, \nabla^k\, \Theta = -\frac{1}{3}\, \lambda \quad \textbf{where} \quad \Theta = 0.$$

For suitable data the expression for Θ admits 2 zeros.

Point sets:

Reparametrize $x(\tau)$ such that $\bar{x}(\bar{\tau}) = x(\tau(\bar{\tau}))$ satisfies $\tilde{g}(\bar{x}', \bar{x}') = 1$.
Write $b = \hat{b} + \zeta\, \dot{x}^\flat$ such that $< \hat{b}, \dot{x} > = 0$. It follows:

on vacuum fields the conformal geodesic equations imply the 'vacuum-adapted conformal geodesic equations'

$$\tilde{\nabla}_{\bar{x}'}\, \bar{x}' = \hat{b}^\sharp, \quad \tilde{\nabla}_{\bar{x}'}\, \hat{b} = \beta^2\, \bar{x}'^\flat,$$

where $\beta^2 \equiv -\tilde{g}^\sharp(\hat{b}, \hat{b}) = (\delta^{ab}\, d_a\, d_b)_* =$ const. along $\bar{x}(\bar{\tau})$,

Thus:

if $\hat{b}$ vanishes at a point, the curve is a $\tilde{g}$-geodesic,

the choice of ζ_* determines the parameterization (slicing),

after a reparameterization conformal geodesics satisfy an equation of third order. Here $\tilde{\nabla}^2_{\bar{x}'}\, \bar{x}' = \beta^2\, \bar{x}$.

The general conformal field equations II.

We call the 'conformal equations based on Weyl connections' in a conformal Gauss gauge the 'general conformal field equations'.

With the 'vacuum expressions for Θ and d_k' (or with the equations governing Θ and d_k) the equations represent a complete system for

$$u = (e^{\mu}{}_{k},\ \hat{\Gamma}_{i}{}^{j}{}_{k},\ \hat{L}_{jk},\ d^{i}{}_{jkl}).$$

A conformal Gauss gauge implies the explicit conditions

$$\dot{x} = e_0 = \partial_\tau, \quad \hat{\Gamma}_{0}{}^{j}{}_{k} = 0, \quad \hat{L}_{0k} = 0.$$

The general conformal field equations thus imply evolution equations of the form

$$\partial_\tau\, e^{\mu}{}_{k} = -\hat{\Gamma}_{k}{}^{l}{}_{0}\, e^{\mu}{}_{l},$$

$$\partial_\tau\, \hat{\Gamma}_{l}{}^{i}{}_{j} = -\hat{\Gamma}_{k}{}^{i}{}_{j}\, \hat{\Gamma}_{l}{}^{k}{}_{0} + g^{i}{}_{0}\, \hat{L}_{lj} - \ g_{j0}\, \hat{L}_{l}{}^{i} + g^{i}{}_{j}\, \hat{L}_{l0} + \Theta\, d^{i}{}_{j0l},$$

$$\partial_\tau\, \hat{L}_{kj} + \hat{\Gamma}_{k}{}^{i}{}_{0}\, \hat{L}_{ij} = d_i\, d^{i}{}_{j0k},$$

$$\nabla_i d^{i}{}_{jkl} = 0.$$

This system implies symmetric hyperbolic 'reduced equations' which preserve the constraints in domains where $\Theta > 0$.

In the conformal Gauss gauge the location of the set $\{\Theta = 0\}$ can be prescribed explictly in terms of the data.