# The motion of the free surface of a liquid

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**Motion of a liquid body in vacuum**

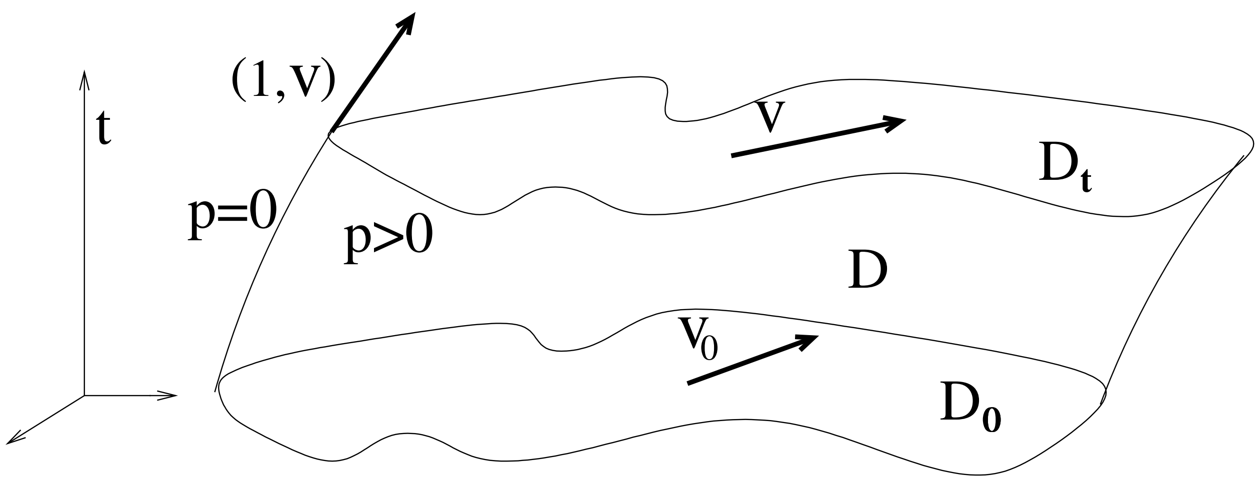
(the ocean or a star)

Incompressible or compressible perfect fluid

Non-relativistic

Without surface tension and gravitation

$v$-velocity, $p$-pressure, $\rho$-density, $t$-time




**Free boundary problem:**

The velocity tells the boundary where to move. The boundary is the zero set of the pressure and the pressure determines the acceleration. (Regularity of the boundary is intimately connected to the regularity of the velocity. )

## Euler's equations

$$\rho\,(\partial_t + V^k\partial_k)v_i = -\partial_i p \quad \text{in } D \quad i{=}1,...,n \quad (1)$$

$$(\partial_t + V^k\partial_k)\rho + \rho\,\text{div}\,V = 0, \quad \text{in } D \quad (2)$$

$$\partial_k{=}\frac{\partial}{\partial x^k},\ V^k{=}v_k,\ V^k\partial_k{=}\textstyle\sum_{k=1}^n V^k\partial_k,\ \text{div}V{=}\partial_k V^k$$

## Equation of state

$$\text{Compressible case: } p = p(\rho), \quad (3)$$

$$p(\bar\rho_0) = 0,\ \bar\rho_0 > 0, \quad p'(\rho) > 0,\ \rho \geq \bar\rho_0 \quad (4)$$

$$\text{Incompressible case: } \rho = \text{constant} \quad (5)$$

## Boundary conditions

$$(\partial_t + V^k\partial_k)|_{\partial D} \in T(\partial D) \quad (6)$$

$$p = 0, \qquad \text{on } \partial D \quad (7)$$

$T(\partial D)$ is the tangent space of the boundary.

## Initial conditions

$$\{x; (0,x) \in D\} = D_0 \quad (8)$$

$$v(0,x) = v_0(x),\ \rho(0,x) = \rho_0(x),\ \text{in } D_0 \quad (9)$$

**Compatibility conditions** Formal power series solution $(\tilde V, \tilde\rho)$, in time of Euler's eq. and initial cond. should satisfy boundary cond.:

$$(\partial_t{+}\tilde V^k\partial_k)^j(\tilde\rho{-}\bar\rho_0)|_{\{0\}\times\partial D_0}{=}0,\ j{=}0,.. \quad (10)$$

**Local Existence?:**
Given a domain $D_0 \subset \mathbf{R^n}$, a vector field $v_0$ and a function $\rho_0$ in $D_0$ satisfying the compatibility conditions (10).
Find a domain $D = \cup_{0\leq t\leq T}\{t\} \times D_t$, $D_t \subset \mathbf{R^n}$, a vector field $v$ and a function $\rho$, depending on $t$ and defined in $D$, such that (1)-(9) hold.

**Local existence for analytic data**
Baouendi-Goulaouic, Nishida
(incompressible irrotational case)

**Instability in Sobolev norms?**
Rayleigh-Taylor Instability
(heavier fluid above lighter)
Ebin's counterexample (when $p<0$, $\nabla_N p>0$).

**Physical condition**

$$\nabla_N p \leq -c_0 < 0, \quad \text{on } \partial D_0, \tag{11}$$

where $\nabla_N = N^k \partial_k$ and $N$ is the exterior normal
Since the pressure of a fluid has to be positive
Needed for local existence in Sobolev Spaces.
Holds in the incompressible irrotational case.

Vorticity: $\operatorname{curl} v_{ij} = \partial_i v_j - \partial_j v_i$
Incompressible fluid: $\operatorname{div} V = 0$
Irrotational fluid: $\operatorname{curl} v = 0$.

**Local existence in Sobolev spaces:**

**I) Incompressible Irrotational case:**
Local existence for Water wave problem:
Yosihara, Nalimov:close to still water in $\mathbf{R}^2$
Wu:in general in $\mathbf{R}^2$ and $\mathbf{R}^3$
(no instability when water wave turns over, physical cond. hold in the irrotational case)

**II) General Incompressible case:**
Ebin-local exist with surface tension(announced)
Christodoulou-L: i) Sobolev norms remain bounded as long as the physical cond. hold, the first order derivatives of the velocity are bounded and the second fundamental form of the free surface is bounded. ii) local *a priori* bounds for Sobolev norms with lower regularity.
L: iii) Local existence assuming physical condition holds initially.

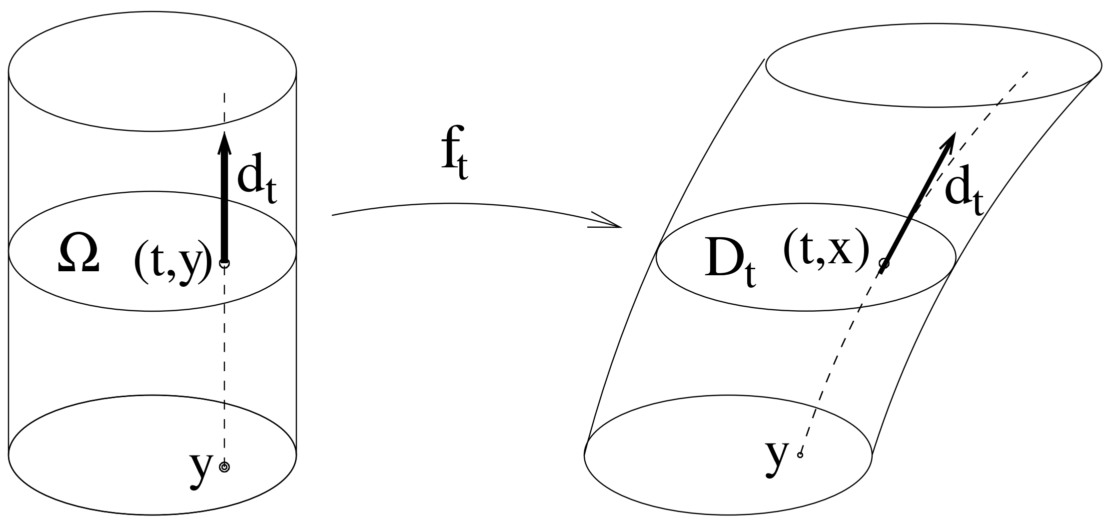
**III) General Compressible case:**
L: Local existence assuming physical condition holds initially.

**IV) Generalizations:**
L: Newtonian gravity, special relativity.
Relativity: Existence in special cases by Rendall, Christodoulou, Friedrich.

**Lagrangian coordinates:** $f_t : y \to x(t,y)$:

$$dx/dt = v(t,x), \quad x(0,y) = f_0(y),\ y \in \Omega$$

Boundary becomes fixed in the $(t,y)$ coord.




| Lagrangian $(t,y)$ | Eulerian $(t,x)$ |
|---|---|
| $[0,T]\times\Omega$ | $D = \cup_{0\le t\le T}\{t\}\times D_t$ |
| $d_t = \partial_t$ | $d_t = \partial_t + V^k\partial_k$ |
| $\partial_k = \frac{\partial y^a}{\partial x^k}\frac{\partial}{\partial y^a}$ | $\partial_k = \frac{\partial}{\partial x^k}$ |

Euler's eq:

$$\rho\, d_t v_i = -\partial_i p,$$
$$d_t\rho + \rho\,\mathrm{div}\,V = 0$$
$$d_t x_i = v_i$$
$$d_t\kappa - \kappa\,\mathrm{div}\,V = 0,$$

where $\kappa = \det(\partial x/\partial y)$, so $\rho = k/\kappa$ where $k = k(y)$ can be chosen to be constant.

## Energies

$E_r(t) = \|v\|_{H^r(D_t)} + \|\rho\|_{H^r(D_t)} + \|\theta\|_{H^{r-2}(\partial D_t)}$

where $\theta_{ij} = \bar{\partial}_i N_j$ is the second fundamental form of $\partial D_t$.

**Energy bound:** $E_r(t) \le C_r(t, c_0^{-1}) E_r(0)$, where $\nabla_N p \le -c_0 < 0$.

**Energy Conservation** $E_0(t) = E_0(0)$ where

$$E_0(t) = \int_{\partial D_t} (|V|^2 + Q(\rho))\rho dx,$$

and $Q(\rho) = 2\int p(\rho)\rho^{-2} d\rho$.

**Proof of Energy conservation:** With $\kappa = \det(\partial x/\partial y)$ we have

$$\int_{D_t} h\rho dx = \int_\Omega h\rho\kappa dy, \qquad d_t(\rho\kappa) = 0,$$

and

$$\frac{d}{dt}E_0 = \int_{\partial D_t} (d_t(|V|^2 + Q(\rho)))\rho dx$$
$$= \int_{D_t} (-2V^i \partial_i p + 2p\rho^{-1} d_t\rho)\, dx =$$
$$-\int_{\partial D_t} 2N_i V^i p dS + \int_{D_t} 2(\partial_i V^i)p + p\rho^{-1} d_t \rho dx = 0$$

**Euler's eq.** With $h(\rho) = \int_{\rho_0}^{\rho} p'(\rho)\rho^{-1}\, d\rho$ the enthalphy we have

$$d_t^2 x^i + \partial_i h = 0, \quad \rho = k/\kappa, \quad h|_{\partial\Omega} = 0$$

where $h = h(\rho)$, $\kappa = \det(\partial x/\partial y)$

**Linearized equations** Consider a family of solutions $x = \bar{x}(t, y, r)$ depending on an extra parameter $r$ and let $\delta x = \partial \bar{x}(t, y, r)/\partial r|_{r=0}$.

$$d_t^2 \delta x^i + \partial_i \delta h - (\partial_k h)\partial_i \delta x^k = 0,$$
$$\delta\rho = -\rho\, \mathrm{div}\delta x, \quad \delta h|_{\partial\Omega} = 0.$$
$$\delta h = h'(\rho)\delta\rho$$

since $[\delta, \partial_i] = -(\partial_i \delta x^k)\partial_k$ and $\delta\kappa = -\kappa \mathrm{div}\delta x$.

**Energy bounds:** $\tilde{E}_r(t) \leq C_r(x, t, c_0^{-1})\tilde{E}_r(0)$ where

$$\tilde{E}_r(t) = \|\delta v\|_{H^r(D_t)} + \|\delta\rho\|_{H^r(D_t)} + \|\delta\theta\|_{H^{r-2}(\partial D_t)}$$

and $\nabla_N h \leq -c_0 < 0$.

**Existence for linearized equations:** Non standard because the higher order operator $-(\partial_k h)\partial_i \delta x^k$ is not elliptic. It is positive because $\nabla_N h < 0$.
**Existence for Euler's eq.:** Follows from existence and tame estimates for the linearized equations using the Nash-Moser technique.

**rewriting the linearized equations:**

$$\ddot{X} + \tilde{A}X = B(X, \dot{X}), \quad \text{div} X|_{\partial\Omega} = 0$$

where $X = \delta x$, $B$ is a bounded operator and $\dot{X} = \hat{L}_{d_t} X$ is a modified Lie derivative:

$$\hat{L}_{d_t} X^i = L_{d_t} X^i + \text{div} V\, X^i = \kappa^{-1} \frac{\partial x^i}{\partial y^a} d_t (\kappa \frac{\partial y^a}{\partial x^k} X^k)$$

that preserves the divergence free condition: $\text{div} \hat{L}_{d_t} X = \hat{d}_t \text{div} X$, where $\hat{d}_t = d_t + \text{div} V$.
$\tilde{A}$ is a positive symmetric operator on vector fields satisfying the boundary condition if the physical condition $\nabla_N p < 0$ hold.

$$\tilde{A}X = -\nabla (h' \rho\, \text{div} X + (\partial_k h) X^k), \quad \nabla^i = \partial_i$$

If $\langle X, Z \rangle = \int_{D_t} X \cdot Z\, dx$, $\text{div} X|_{\partial\Omega} = \text{div} Z|_{\partial\Omega} = 0$:

$$\langle X, \rho\, \tilde{A} Z \rangle = \int_{D_t} \text{div}(\rho X) \text{div}(\rho Z) h' dx + \int_{\partial D_t} X_N Z_N (-\nabla_N p) dS, \quad X_N = X \cdot N$$

**Energy** $\tilde{E}_0 = \langle \dot{X}, \rho \dot{X} \rangle + \langle X, \rho \tilde{A} X \rangle$

$$\tilde{E}_r = \|\dot{X}\|_{H^r(\Omega)} + \|\text{div} X\|_{H^r(\Omega)} + \|X_N\|_{H^r(\partial\Omega)}$$

**Orthogonal projection onto divergence free vector fields**

$$PX = X - \nabla q, \quad \triangle q = \text{div}X, \;\; q|_{\partial\Omega} = 0$$

Decompose: $X = X_0 + X_1$, where $X_0 = PX$.

$$\hat{d}_t^2 \text{div}X - \triangle(p' \text{div}X) = \triangle(X^k \partial_k h) + \text{div } B,$$

with Dirichlet boundary cond. $\text{div}X|_{\partial\Omega} = 0$

$$AX = P\tilde{A}X = P(-\nabla(X^k \partial_k h))$$

since the projection of the gradient of a function that vanishes on the boundary vanishes.

$$\ddot{X}_0 + AX_0 = -AX_1 - PB_2(X_1, \dot{X}_1) + PB(X, \dot{X})$$

using that $[\hat{L}_{d_t}, P]X = O(X)$. Here $A$ is symmetric and positive:

$$\langle X, AZ \rangle = \int_{\partial D_t} X_N Z_N (-\nabla_N h) dS, \quad \text{div}X = \text{div}Z = 0$$

**Energies** $E_0 = \langle \dot{X}_0, \dot{X}_0 \rangle + \langle X_0, AX_0 \rangle$.

**Existence and estimates for the divergence free equation:**

$$\ddot{X} + AX = F, \quad \text{div}X = \text{div}F = 0$$

$$E_r = \|\dot{X}\|_{H^r(\Omega)} + \|X_N\|_{H^r(\partial\Omega)}$$

$$E_r(t) \le C_r(E_r(0) + \int_0^t \|F\|_{H^r(\Omega)} d\tau)$$

Replace $A$ by a a sequence of bounded operators $A^\varepsilon$ for which existence is know and such that we uniformly have the same energy estimates as $\varepsilon \to 0$.

$$\begin{aligned} A^\varepsilon X &= -P(\chi_\varepsilon(d)\nabla h d^{-1} X^k \partial_k d) \\ &= P(\chi_\varepsilon'(d)(\nabla d) h d^{-1} X^k \partial_k d) \end{aligned}$$

where $d = d(y) = \text{dist}(y, \partial\Omega)$, $\chi_\varepsilon(d) = \chi(d/\varepsilon)$. Here $\chi(s) = 1$, when $s \ge 1$, $\chi(s) = 0$, when $s \le 0$, and $\chi'(s) \ge 0$.