Global Lorentzian Geometry and the Einstein Equations

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Part I

- Causal theory
- Geometry of smooth null hypersurfaces
- Maximum Principle for smooth null hypersurfaces

Part II

- Achronal boundaries
- C^0 null hypersurfaces
- Maximum Principle for C^0 null hypersurfaces
- The null splitting theorem

Part III - Applications

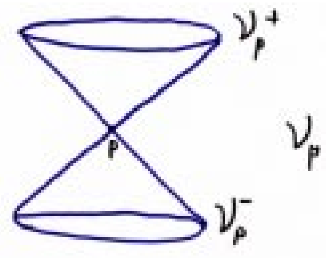
- Uniqueness results for asymptotically de Sitter and asymptotically flat solutions of the vacuum Einstein equations
- Results on the topology of asymptotically de Sitter solutions of the Einstein equations

Elements of Lorentzian Geometry

$$
\begin{aligned}
M^{n+1} &= \text{smooth Lorentzian manifold} \\
&= \text{smooth manifold equipped with metric} \\
&\quad\; g = \langle\, , \,\rangle \text{ having signature } (- + \cdots +)
\end{aligned}
$$

The null cone $\mathcal{V}_p$ at $p \in M$ is the set,

$$\mathcal{V}_p = \{X \in T_pM; \langle X, X\rangle = g_{ij}X^iX^j = 0\}$$



We always assume M is time orientable , i.e. that the assignment of a past and future cone, $\mathcal{V}_p^-$ and $\mathcal{V}_p^+$, can be made in a continuous manner on M.

spacetime = time oriented Lorentzian manifold

Let,

$$\nabla = \text{Levi-Civita connection}$$

For vector fields $X = X^a$, $Y = Y^b$,

$$\nabla_X Y = X^a \nabla_a Y^b$$

Riemann curvature tensor. For vector fields X, Y, Z,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The components are determined by,

$$R(\partial_i, \partial_j)\partial_k = R^\ell{}_{kij}\partial_\ell$$

The Ricci tensor and scalar curvature are obtained by tracing,

$$R_{ij} = R^\ell{}_{i\ell j} \quad \text{and} \quad R = g^{ij} R_{ij}$$

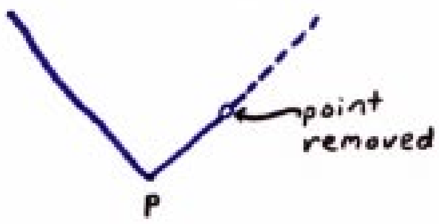
Past and Futures:

Def. For $p \in M$,

$$\begin{aligned} I^+(p) &= \text{timelike future of } p \\ &= \{q \in M : \exists \textit{ future directed} \text{ timelike curve from } p \text{ to } q\} \end{aligned}$$

$$\begin{aligned} J^+(p) &= \text{causal future of } p \\ &= \{q \in M : \exists \textit{ future directed} \text{ causal curve from } p \text{ to } q\} \end{aligned}$$

Note: $I^+(p)$ is always open, but $J^+(p)$ need not be closed.

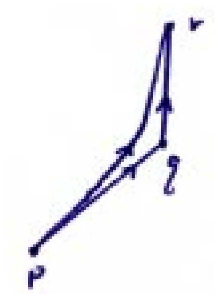


Def. For $A \subset M$,

$$I^+(A) = \{q \in M : \exists \text{ *future directed* timelike curve from some } p \in A \text{ to } q\}$$
$$= \cup_{p \in A} I^+(p) \qquad \text{(always open)}$$

$$J^+(A) = \{q \in M : \exists \text{ *future directed* causal curve from some } p \in A \text{ to } q\}$$
$$= \cup_{p \in A} J^+(p)$$

Prop. $q \in J^+(p)$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$, etc.



Prop. *If $q \in J^+(p) \setminus I^+(p)$ then any causal curve γ from p to q must be a null geodesic.*

<u>*Note:*</u> $I^-(p)$, $J^-(p)$, $I^-(A)$, $J^-(A)$ defined time-dually.

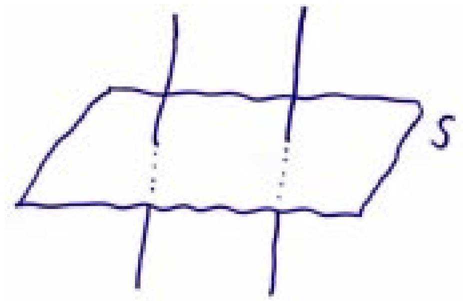
Global hyperbolicity:

Def. *Strong Causality* holds at $p \in M$ provided there are arbitrarily small neighborhoods U of p such that any causal curve γ which starts in, and leaves, U never returns to U.

Def. M is globally hyperbolic provided

- M is strongly causal
- The sets $J^+(p) \cap J^-(q)$ are compact $\forall p, q \in M$

Def. A Cauchy surface for M is an achronal C^0 hypersurface S in M which is met by every inextendible causal curve in M.



Comment: Equivalently, an achronal hypersurface S is Cauchy provided $D(S) = M \iff H(S) = \emptyset$

Prop. M is globally hyperbolic iff M admits a Cauchy surface.

Prop. If S is a Cauchy surface for M then M is homeomorphic to $\mathbb{R} \times S$.

(Moreover the homeomorphism can be arranged so that $\{t\} \times S$ is Cauchy $\forall t$.)

Prop. If S is a compact achronal hypersurface in a globally hyperbolic spacetime M then S must be a Cauchy surface for M.

Prop. If M is globally hyperbolic then

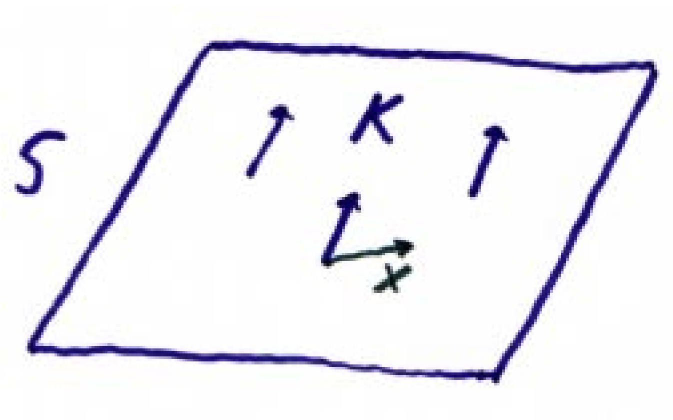
- $J^{\pm}(A)$ are closed $\forall A \subset M$ compact.
- $J^{+}(A) \cap J^{-}(B)$ is compact $\forall A, B \subset M$ compact.

Geometry of Null Hypersurfaces

Def. A smooth null hypersurface in (M, g) is a smooth co-dim one submanifold S of M, such that the pullback of g to S is degenerate.

Such an S admits a smooth future directed null tangent vector field K such that

$$[K_p]^{\perp} = T_pS \qquad \forall p \in S$$



Note:

- Every vector X tangent to S, and not a multiple of K, is spacelike.
- K is unique up to a positive pointwise scale factor.

Prop. *The integral curves of K, when suitably parameterized, are null geodesics (and are called the null generators of S).*

Proof: Suffices to show:

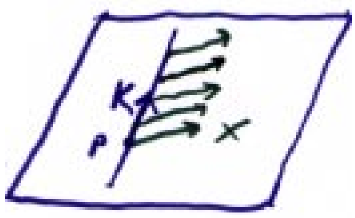
$$\nabla_K K = \lambda K$$

This follows by showing at each $p \in S$,

$$\nabla_K K \perp T_p S\,, \quad \text{i.e.,} \quad \langle \nabla_K K, X\rangle = 0 \quad \forall X \in T_p S$$

Extend $X \in T_p S$ by making it invariant under the flow generated by K,

$$[K, X] = \nabla_K X - \nabla_X K = 0$$



X remains tangent to S, so along the flow line through p,

$$\langle K, X\rangle = 0$$

Differentiating,

$$\begin{aligned} K\langle K, X\rangle &= \langle \nabla_K K, X\rangle + \langle K, \nabla_K X\rangle = 0 \\ \langle \nabla_K K, X\rangle &= -\langle K, \nabla_X K\rangle = -\frac{1}{2} X\langle K, K\rangle = 0. \end{aligned}$$

QED

Null Weingarten Map/Null 2nd Fundamental Form.

One works mod K: For $X, Y \in T_pS$,

$$X = Y \text{ mod } K \iff X - Y = \lambda K$$

Let $\overline{X}$ denote equivalence class of $X \in T_pM$ and let,

$$T_pX/K = \{\overline{X} : X \in T_pM\}$$

Then,

$$TS/K = \cup_{p\in S} T_pS/K$$

is a rank $n-1$ vector bundle over S ($n = \dim S$).

Positive definite metric on TS/K:

$$\begin{aligned} h : T_pS/K \times T_pS/K &\to \mathbb{R} \\ h(\overline{X}, \overline{Y}) &= \langle X, Y \rangle \end{aligned}$$

Well-defined: $X' = X$ mod K, $Y' = Y$ mod $K \Rightarrow$

$$\begin{aligned} \langle X', Y' \rangle &= \langle X + \alpha K, Y + \beta K \rangle \\ &= \langle X, Y \rangle + \beta \langle X, K \rangle + \alpha \langle K, Y \rangle + \alpha\beta \langle K, K \rangle \\ &= \langle X, Y \rangle \end{aligned}$$

Weingarten Map:

$$b : T_pS/K \rightarrow T_pS/K$$

$$b(\overline{X}) = \overline{\nabla_X K}$$

Well-defined: $X' = X \mod K \Rightarrow$

$$\begin{aligned} \nabla_{X'} K &= \nabla_{X+\alpha K} K \\ &= \nabla_X K + \alpha \nabla_K K = \nabla_X K + \alpha \lambda K \\ &= \nabla_X K \quad \text{mod } K \end{aligned}$$

Second Fundamental Form:

$$B : T_pS/K \times T_pS/K \rightarrow \mathbb{R}$$

$$\begin{aligned} B(\overline{X}, \overline{Y}) &= h(b(\overline{X}), \overline{Y}) \\ &= \langle \nabla_X K, Y \rangle \end{aligned}$$

Prop. *B is symmetric, $B(\overline{X}, \overline{Y}) = B(\overline{Y}, \overline{X}), \forall \overline{X}, \overline{Y} \in T_pS/K$, and hence b is self-adjoint.*

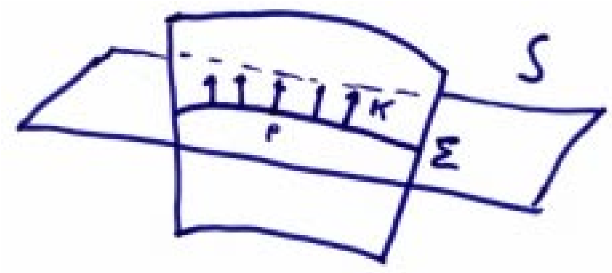
Proof. Extend X, Y to vector fields tangent to S near p. Using $X\langle K, Y\rangle = 0$ and $Y\langle K, X\rangle = 0$,

$$\begin{aligned} B(\overline{X}, \overline{Y}) &= \langle \nabla_X K, Y \rangle = -\langle K, \nabla_X Y \rangle \\ &= -\langle K, \nabla_Y X \rangle + \langle K, [X, Y] \rangle \\ &= \langle \nabla_Y K, X \rangle = B(\overline{Y}, \overline{X}) \end{aligned}$$

Null mean curvature (expansion scalar):

$$\begin{aligned}\theta &= \operatorname{tr} b \\ &= \operatorname{div} K \quad \text{(essentially)}\end{aligned}$$

Let Σ be the intersection of S with a hypersurface in M which is transverse to K near $p \in S$; Σ will be an $n-1$ dimensional spacelike submanifold of M.



Let $\{e_1, e_2, \cdots, e_{n-1}\}$ be an orthonormal basis for $T_p\Sigma$ in the induced metric. Then $\{\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_{n-1}\}$ is an orthonormal basis for T_pS/K.

Hence,

$$\begin{aligned}\theta = \operatorname{tr} b &= \sum_{i=1}^{n-1} h(b(\overline{e}_i), \overline{e}_i) \\ &= \sum_{i=1}^{n-1} \langle \nabla_{e_i} K, e_i \rangle \\ &= \operatorname{div}_\Sigma K \quad \text{at } p\end{aligned}$$

Thus, θ measures the expansion of the null generators of S towards the future.

$\theta > 0$ $\qquad\qquad$ $\theta < 0$

Effect of scaling K:

Prop. $\widetilde{K} = fK \Rightarrow b_{\widetilde{K}} = f\, b_K$, and hence $\theta_{\widetilde{K}} = f\,\theta_K$

Proof:

$$\nabla_X \widetilde{K} = \nabla_X(fK) = X(f)K + f\nabla_X K = f\nabla_X K \quad \mathrm{mod} K$$

Note: In particular, the Weingarten map $b = b_K$ at a point $p \in S$ is uniquely determined by the value of K at p.

Comparison theory.

Let $\eta : I \to M$, $s \to \eta(s)$, be an affinely parameterized null geodesic generator of S, and let

$$b(s) = b_{\eta'(s)}$$

be the null Weingarten map at $\eta(s)$ wrt the null tangent vector $\eta'(s)$.

The family of Weingarten maps $b = b(s)$ along η obeys the Ricatti equation,

$$b' + b^2 + R = 0, \qquad ' = \nabla_{\eta'}$$

where, by def.,

$$b'(\overline{X}) = b(\overline{X})' - b(\overline{X}'), \quad \text{(and } (\overline{Y})' = \overline{Y'}\text{)}$$

$$R(\overline{X}) = \overline{R(X, \eta')\eta'}\,.$$

Proof:

Fix $p = \eta(s_0)$ on η, and scale K so that in a neighborhood of p,

(i) K is geodesic, $\nabla_K K = 0$.

(ii) $K = \eta'$ along η.

Extend $X \in T_pS$ near p by making it invariant under the flow generated by K,

$$[K, X] = \nabla_K X - \nabla_X K = 0\,.$$

Then,

$$R(X,K)K = \nabla_X \nabla_K K - \nabla_K \nabla_X K - \nabla_{[X,K]} K = -\nabla_K \nabla_K X\,,$$

i.e., along η, X satisfies,

$$X'' = -R(X, \eta')\eta'\,.$$

Thus,

$$\begin{aligned} b'(\overline{X}) &= \overline{\nabla_X K}' - b(\overline{\nabla_K X}) = \overline{\nabla_K X}' - b(\overline{\nabla_X K}) \\ &= \overline{X''} - b(b(\overline{X})) = -\overline{R(X,\eta')\eta'} - b^2(\overline{X}) \\ &= -R(\overline{X}) - b^2(\overline{X}) \end{aligned}$$

QED

By tracing, we obtain along η that $\theta = \theta(s)$ obeys,

$$\frac{d\theta}{ds} \quad = \quad -\text{Ric}\,(\eta', \eta') - \text{tr}\, b^2 \,,$$

or,

$$\boxed{\frac{d\theta}{ds} = -\text{Ric}\,(\eta', \eta') - \sigma^2 - \frac{1}{n-2}\theta^2}$$

Raychaudhuri's equation

where σ is the *shear* scalar, $\sigma^2 = \text{tr}\,\widehat{b}^2$, $\widehat{b} = b - \frac{1}{n-1}\theta \cdot \text{id}$.

Prop. *Let S be a smooth null hypersurface in a space-time M which obeys the null energy condition,*

$$\text{Ric}\,(X, X) \geq 0 \quad \forall \textit{ null vectors } X \,.$$

Then, if the null generators of S are future geodesically complete, S has nonnegative null mean curvature, $\theta \geq 0$.

Proof. Suppose $\theta < 0$ at $p \in S$. Let $s \to \eta(s)$ be the null generator of S passing through $p = \eta(0)$.

Let $b(s) = b_{\eta'(s)}$, and take $\theta = \text{tr}\ b$. By invariance of sign under scaling, $\theta(0) < 0$.

By Raychaudhuri's equation and the NEC,

$$\frac{d\theta}{ds} \leq -\frac{1}{n-1}\theta^2 \,,$$

and hence $\theta < 0$ for $s > 0$. Dividing through by θ^2 gives,

$$\frac{d}{ds}\left(\frac{1}{\theta}\right) \geq \frac{1}{n-1} \,,$$

which implies $1/\theta \to 0$, i.e., $\theta \to -\infty$ in finite affine parameter time, $\rightarrow\leftarrow$.

Re: Hawking area theorem; cf., Chruściel, Delay, G. Howard (2001).

Totally geodesic null hypersurfaces.

By def., a smooth null hypersurface S is *totally geodesic* iff $B \equiv 0$ (or, equivalently, iff $\theta = \sigma = 0$).

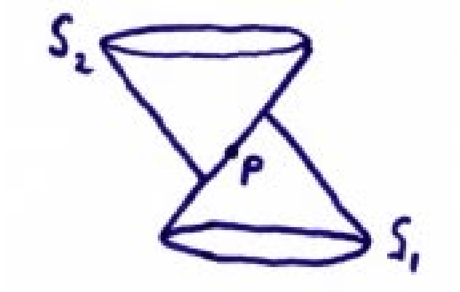
Prop. *A null hypersurfaces S is totally geodesic iff geodesics starting tangent to S remain in S.*

Ex. Null hyperplanes in Minkowski space, the event horizon in Schwarzschild are totally geodesic.

Maximum Principle for Smooth Null hypersurfaces.

Theorem. *Suppose*

- S_1 *and* S_2 *are smooth null hypersurfaces in* M*.*
- S_1 , S_2 *meet at* $p \in M$*, with* S_2 *to the future side of* S_1 *near* p*.*



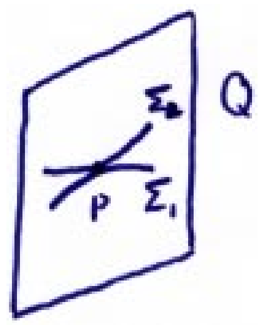
- $\theta_2 \le 0 \le \theta_1$.

Then S_1 *and* S_2 *coincide near* p*, and* $\theta_1 = \theta_2 = 0$*.*

Proof:

S_1 and S_2 have a common null direction at p. Let Q be a timelike hypersurface in M passing through p and transverse to this direction. Consider the intersections,

$$\Sigma_1 = S_1 \cap Q\,, \qquad \Sigma_2 = S_2 \cap Q$$



Σ_1 and Σ_2 are *spacelike* hypersurfaces in Q, with Σ_2 to the future of Σ_1 near p.

Express Σ_1 and Σ_2 as graphs over a fixed hypersurface V in Q,

$$\Sigma_1 = \text{ graph}\,(u_1)\,, \qquad \Sigma_2 = \text{ graph}\,(u_2)$$

Let,

$$\theta(u_i) = \theta_i|_{\Sigma_i = \text{ graph}\,(u_i)}\,, \quad i = 1, 2$$

By a computation,

$$\theta(u_i) = H(u_i) + \text{ l.o.t.}$$

where $H =$ mean curvature operator on spacelike graphs over V in Q. Thus θ is a second order quasi-linear elliptic operator.

We have:

- $u_1 \leq u_2$, and $u_1(p) = u_2(p)$.
- $\theta(u_2) \leq 0 \leq \theta(u_1)$.

By the strong maximum principle, $u_1 = u_2$.

Thus, S_1 and S_2 agree near p in Q. Now, vary Q to get agreement on a neighborhood.

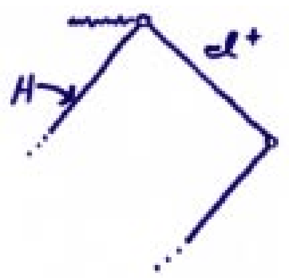
C^0 Null hypersurfaces

In GR, the null hypersurfaces of interest, e.g. horizons of various sorts, are not smooth in general.

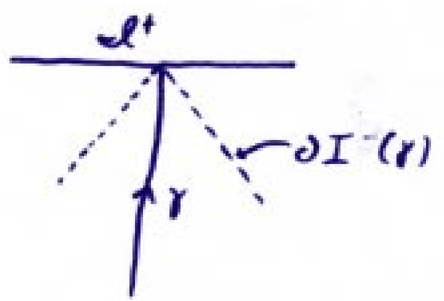
Such hypersurfaces often arise as the null portions of achronal boundaries, i.e., boundaries of pasts/futures,

$$A \subset M, \qquad \partial I^{\pm}(A) = \text{ achronal boundary.}$$

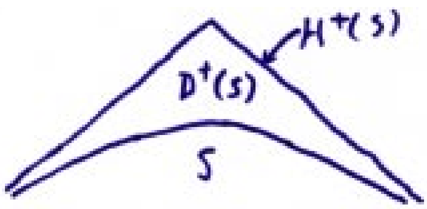
- Black hole event horizon: $H = \partial I^{-}(\mathcal{J}^{+}) \cap M$



- Observer horizons: $\partial I^{-}(\gamma)$

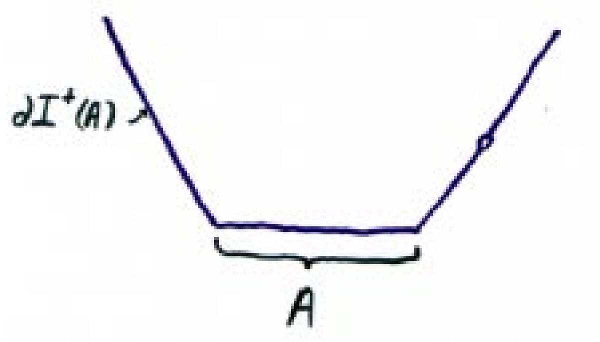


- Cauchy horizons: $H^{+}(S) = \partial I^{-}(D^{+}(S)) \cap J^{+}(S)$



Achronal boundaries.

Def. An achronal boundary is a set of the form $\partial I^+(A)$ (or $\partial I^-(A)$).



Prop. An achronal boundary $\partial I^+(A)$, if nonempty, is a closed achronal C^0 hypersurface in M.

Discussion of proof:

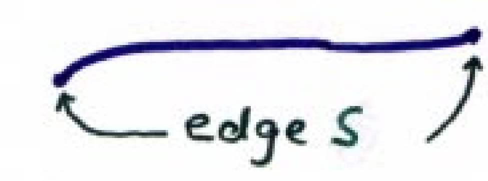
Lemma. *If $p \in \partial I^+(A)$ then $I^+(p) \subset I^+(A)$, and similarly, $I^-(p) \subset M \setminus \overline{I^+(A)}$.*

Pf: $q \in I^+(p) \Rightarrow p \in I^-(q)$. Since $I^-(q)$ is a nbd of p, and p is on the boundary of $I^+(A)$, $I^-(q) \cap I^+(A) \neq \emptyset$, and hence $q \in I^+(A)$.

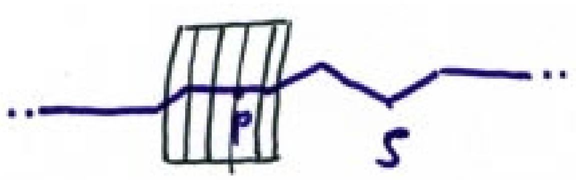
Since $I^+(A)$ does not meet $\partial I^+(A)$, it follows from the lemma that $\partial I^+(A)$ is achronal.

It also follows from the lemma that $\partial I^+(A)$ is edgeless.

Def. The edge of a closed achronal set $S \subset M$ is the set of points $p \in S$ such that every neighborhood U of p, contains a timelike curve from $I^-(p,U)$ to $I^+(p,U)$ that does *not* meet S.

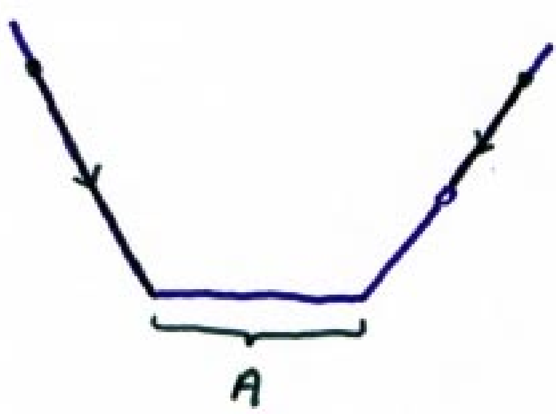


Prop. *A closed achronal edgeless set $S \subset M$ is a C^0 hypersurface in M.*

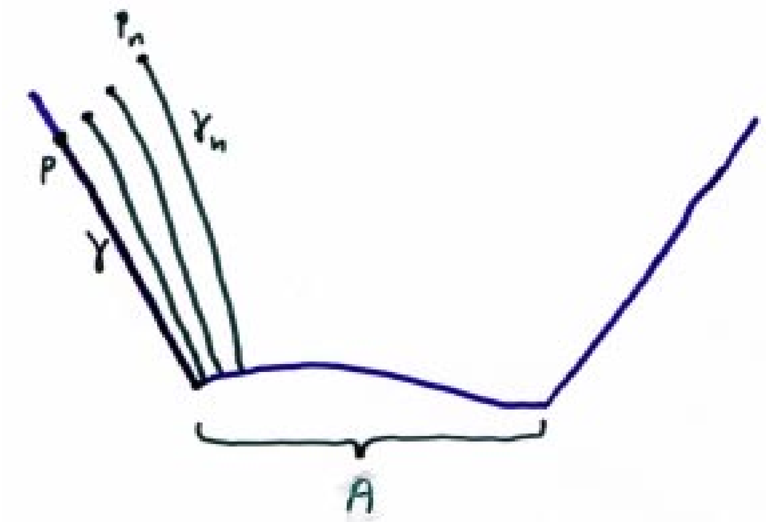


As a corollary, achronal boundaries are C^0 hypersurfaces.

Prop. *Let $A \subset M$ be closed. Then each $p \in \partial I^+(A) \setminus A$ lies on a null geodesic contained in $\partial I^+(A)$, which either has a past end point on A, or else is past inextendible in M.*



Proof.



Choose $\{p_n\} \subset I^+(A)$ such that $p_n \to p$, and let γ_n be a past directed timelike curve from p_n to A. By Ascoli, and passing to a subsequence, $\{\gamma_n\}$ converges to a past directed causal curve $\gamma \subset \partial I^+(A)$ from p. Since γ is both causal and achronal, it must be a null geodesic.

Each γ_n is past inextendible in $M \setminus A$, and hence so is γ. Thus γ either has a past end point on A or is past inextendible in M.

C^0 null hypersurfaces.

Thus sets of the form

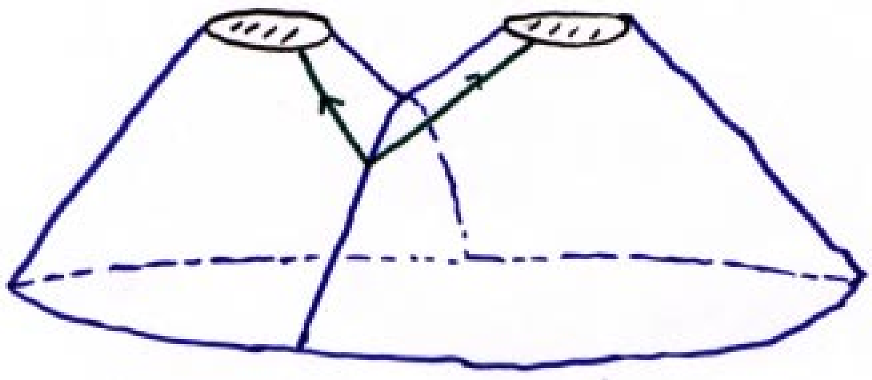
$$S = \partial I^+(A) \setminus A, \quad \text{resp.,} \quad S = \partial I^-(A) \setminus A\,,$$

with A closed, are achronal C^0 hypersurfaces, ruled by null geodesics which are past, resp. future, inextendible in S.

Def. A C^0 *future* null hypersurface is a locally achronal C^0 hypersurface S, which is ruled by null geodesics that are *future* inextendible in S.

Ex. $S = \partial I^-(A) \setminus A$, $A \subset M$ closed.

Ex. $M =$ Minkowski 3-space, $A =$ two disjoint spacelike disks in $t = 0$. Then $S = \partial I^-(A) \setminus A$ is a C^0 future null hypersurface in M



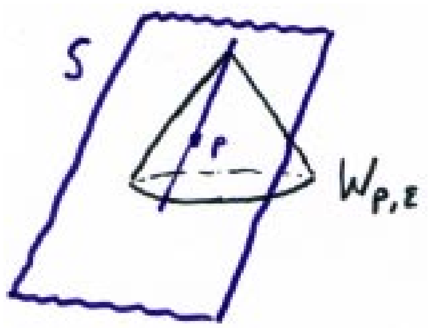
A C^0 *past* null hypersurface is defined time-dually: it is ruled by null geodesics that are *past* inextendible within the hypersurface.

Mean curvature inequalities for C^0 null hypersurfaces.

C^0 null hypersurfaces do not have null mean curvature in the classical sense, but may obey null mean curvature inequalities in a *support sense*.

Def. Let S be a C^0 future null hypersurface. S has null mean curvature $\theta \geq 0$ *in the support sense* provided $\forall p \in S$, and $\forall \epsilon > 0$, there exists a smooth (C^2) null hypersurface $W_{p,\epsilon}$ such that

- $W_{p,\epsilon}$ is a past support hypersurface for S at p.
- $\theta_{p,\epsilon}(p) \geq -\epsilon$.



(For this definition, it is assumed that the null vectors have been uniformly scaled, e.g., have unit length wrt a background Riemannian metric.)

Note: If S is smooth then $\theta \geq 0$ in the usual sense.

Ex. M = Minkowski space, $S = \partial I^+(p)$. S is a C^0 future null hypersurface having $\theta \geq 0$ in the support sense.

If S is a C^0 *past* null hypersurface, one defines $\theta \leq 0$ *in a support sense* in an analogous manner in terms of *future* support hypersurfaces.

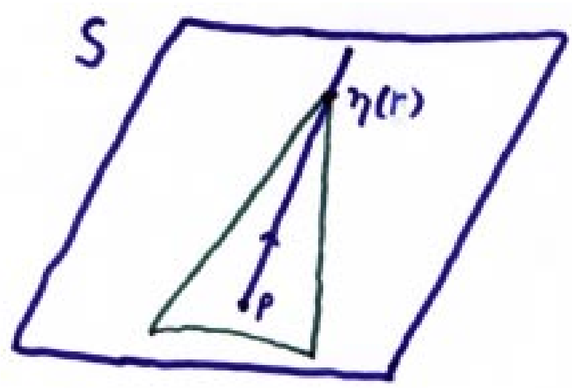
Prop. *Let S be a C^0 future null hypersurface in M. Suppose,*

- *M obeys the null energy condition.*
- *The null generators of S are future geodesically complete.*

Then $\theta \geq 0$ in the support sense.

Proof: WLOG, may assume S is achronal. Given $p \in S$, let $\eta : [0, \infty) \to S \subset M$, $s \to \eta(s)$, be a null generator of S from $p = \eta(0)$.

For any $r > 0$, consider small pencil of past directed null geodesics from $\eta(r)$. Forms a smooth (caustic free) null hypersurface $W_{p,r}$ containing $\eta|_{[0,r]}$, which is a lower support hypersurface for S at p.



Let $\theta = \theta(s)$, $0 \leq s \leq r$, be the null mean curvature of $W_{p,r}$ along $\eta|_{[0,r]}$.

By Raychaudhuri and NEC we have,

$$\frac{d\theta}{ds} \leq -\frac{1}{n-1}\theta^2,$$

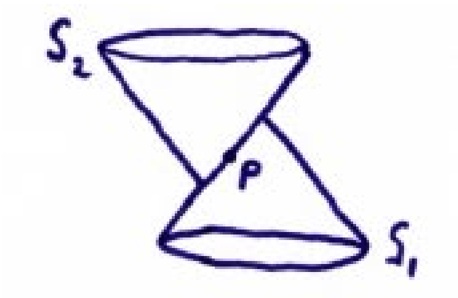
Together with $\theta(r) = -\infty$ gives,

$$\theta(0) \geq -\frac{n-1}{r}$$

Maximum principle for C^0 null hypersurfaces.

Theorem. *Suppose*

- *S_1 is a C^0 future null hypersurface, and S_2 is a C^0 past null hypersurface in M.*
- *S_1 , S_2 meet at $p \in M$, with S_2 to the future side of S_1 near p.*



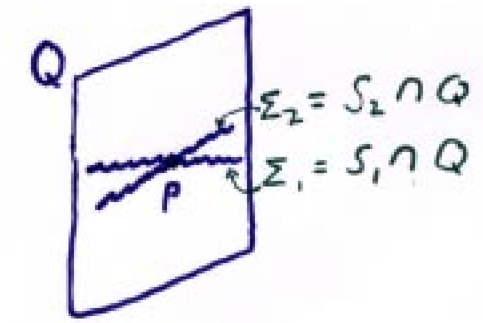
- *$\theta_2 \leq 0 \leq \theta_1$ in the support sense.*

Then S_1 and S_2 coincide near p, and form a smooth *null hypersurface with $\theta = 0$.*

Comments on the proof:

Although there are some technical issues, the proof proceeds essentially as in the smooth case.

Can show p is an interior point of a null generator common to both S_1 and S_2 near p. As before, intersect S_1 and S_2 with a timelike hypersurface Q through p, transverse to this generator.



Σ_1 and Σ_2 will be C^0 *spacelike* hypersurfaces in Q, with Σ_2 to the future of Σ_1 near p.

Can express Σ_1 and Σ_2 as graphs over a fixed hypersurface in Q,

$$\Sigma_1 = \text{graph}\,(u_1)\,, \quad \Sigma_2 = \text{graph}\,(u_2)$$

One has:

- $u_1 \le u_2$, and $u_1(p) = u_2(p)$.
- $\theta(u_2) \le 0 \le \theta(u_1)$ in the support sense.

Need a suitable weak version of the strong maximum principle: Andersson, Howard, G. ('98, Comm. Pure Appl. Math.)

For further details, see: G., Ann. Henri Poincaré **1** (2000) 543.

The Null Splitting Theorem

Lines in Spacetimes.

A *timelike line* is an inextendible timelike geodesic each segment of which is maximal.

The standard Lorentzian splitting theorem describes the rigidity of spacetimes containing a timelike line:

Theorem. *Suppose*

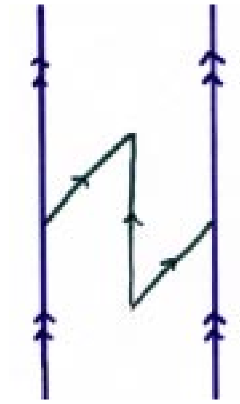
- M *is timelike geodesically complete.*
- M *obeys the strong energy condition,* $\mathrm{Ric}(X,X) \geq 0$, *for timelike* X.
- M *has a timelike line.*

Then M *splits isometrically along the line, i.e.,* (M,g) *is isometric to* $(\mathbb{R} \times V, -dt^2 \oplus h)$, *where* (V,h) *is a complete Riemannian manifold.*

Comment:

- Precise analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry.
- Recall, posed as a problem by Yau in the early 80's as an approach to removing the genericity assumptions in the Hawking-Penrose singularity theorems.

A null line in a spacetime M is an inextendible null geodesic which is *achronal*, i.e. no two points can be joined by a timelike curve. (Thus each segment of a null line is maximal.)



- Global condition.

- Null lines arise naturally in causal arguments: E.g., recall sets of the form,

$$\partial I^{\pm}(A) \setminus A\,, \qquad A \text{ closed}$$

  are ruled by null geodesics which must be achronal.
- Null lines have arisen in many situations, e.g., the Hawking-Penrose singularity theorems, topological censorship, Penrose-Sorkin-Woolgar approach to positive mass, and related results of Gao-Wald on gravitational time delay, etc.
- Examples: Minkowski space, de Sitter, anti-de Sitter, Schwarzschild

One expects some rigidity in spacetimes which contain a null line and which obey the null energy condition.

The NEC tends to focus congruences of null geodesics, which can lead to the occurence of null conjugate points. A null geodesic containing a pair of null conjugate points can't be achronal.

Theorem (*Null Splitting Theorem*). *Suppose,*

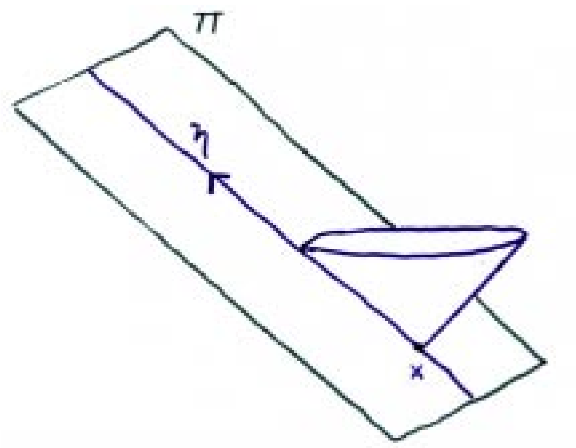
- M *is null geodesically complete.*
- M *obeys the null energy condition,* Ric$(X, X) \geq 0$ *for all null* X.
- M *contains a null line* η.

Then η *is contained in a smooth closed achronal totally geodesic null hypersurface* S.

(1) Ex. Minkowski space - each null geodesic is contained in a unique null hyperplane.

(2) The "splitting" is in S: $B = 0 \iff \theta = \sigma = 0 \iff$ metric h on TS/K is invariant under flow generated by K.

(3) The proof is an application of the maximum principle for C^0 null hypersurfaces. To motivate, consider the situation in Minkowski space:



$$\begin{aligned} \Pi &= \lim_{x \to -\infty} \partial I^{+}(x) = \partial I^{+}(\eta) \\ &= \lim_{y \to \infty} \partial I^{-}(y) = \partial I^{-}(\eta) \end{aligned}$$

Proof: Set,

$$S_+ = \partial I^+(\eta)\,, \qquad S_- = \partial I^-(\eta)$$

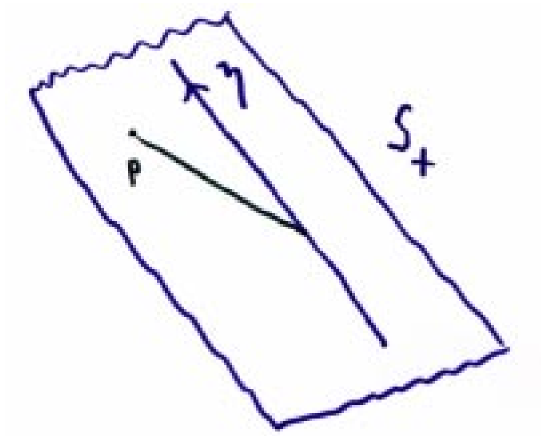
Since η is achronal, $\eta \subset S_+ \cap S_-$.

Claim. S_+ is a C^0 past null hypersurface whose null generators are past inextendible in M. (Similarly for S_-.)

Pf: As an achronal boundary, S_+ is an achronal C^0 hypersurface.

Now, for simplicity assume M is strongly causal. Then η is closed as a subset of M.

By property of achronal boundaries, each point $p \in S_+ \setminus \eta$ is on a null geodesic which is either past inextendible in M or else has past endpoint on η. The latter is impossible:



This violates the achronality of S_+.

Claim. The null mean curvature of S_- and of S_+ satisfy,

$$\theta_+ \le 0 \le \theta_- \quad \text{in the support sense.}$$

Pf: By the completeness assumption, and the previous claim, the generators of S_+ are past complete, and the generators of S_- are future complete. Thus, the claim follows from a previous proposition.

At each point of the intersection $p \in S_+ \cap S_-$, S_+ lies locally to the future of S_-. Thus be the maximum principle $S+$ and S_- agree near p, and form a smooth null hypersurface with vanishing null mean curvature.

It follows that $S_+ \cap S_-$ is both open and closed in S_+ and in S_-. Thus,

$$S_+ = S_+ \cap S_- = S_-\,,$$

and $S = S_+ = S_-$ is a smooth null hypersurface with $\theta = 0$.

Raychaudhuri's equation,

$$\frac{d\theta}{ds} = -\mathrm{Ric}\,(\eta', \eta') - \sigma^2 - \frac{1}{n-1}\theta^2$$

and the NEC now imply that S is totally geodesic.

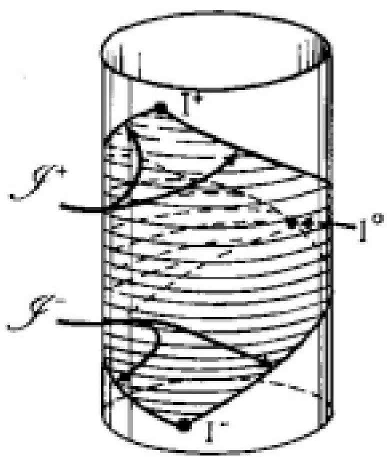
Note: With regard to the completeness assumption, the proof only requires that the generators of $\partial I^+(\eta)$ be past complete and the generators of $\partial I^-(\eta)$ be future compete.

Applications

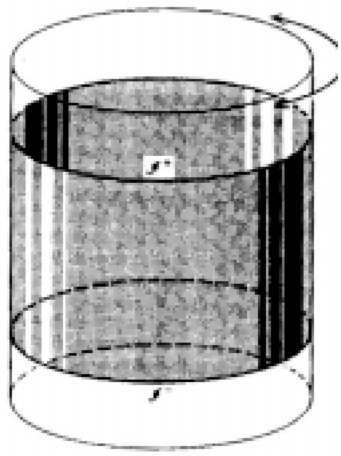
Consider here some global results for solutions to the Einstein equations with prescribed asymptotics.

Use Penrose's notion of conformal infinity to treat the asymptotics.

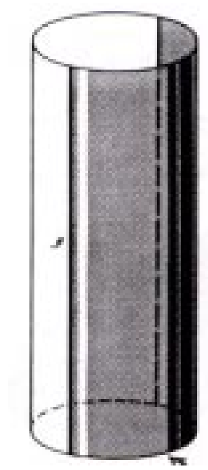
Based on the conformal imbeddings of Minkowski space, de Sitter space and anti-de Sitter space into the Einstein static universe:



Minkowki



de Sitter



anti-de Sitter

We are going to focus primarily on spacetimes which are asymptotically de Sitter, i.e., for which the conformal boundary $\mathcal{J}$ is spacelike.

Def. (M, g) is a spacetime of *de Sitter type* provided there exists a *smooth* spacetime-with-boundary $(\tilde{M}, \tilde{g})$ and a *smooth* function Ω on $\tilde{M}$ such that

- M is the interior of $\tilde{M}$; hence $\tilde{M} = M \cup \mathcal{J}$, $\mathcal{J} = \partial \tilde{M}$.
- $\tilde{g} = \Omega^2 g$, where (i) $\Omega > 0$ on M, and (ii) $\Omega = 0$, $d\Omega \neq 0$ along $\mathcal{J}$.
- $\mathcal{J}$ is spacelike.

$\mathcal{J}$ decomposes into two disjoint sets,

$$\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^-$$

where, $\mathcal{J}^+ \subset I^+(M, \tilde{M})$ and $\mathcal{J}^- \subset I^-(M, \tilde{M})$.

Def. A spacetime M of de Sitter type is *asymptotically simple* provided each inextendible null geodesic in M has a future end point on $\mathcal{J}^+$ and a past end point on $\mathcal{J}^-$.

Ex. De Sitter space, which can be expressed in global coordinates as,

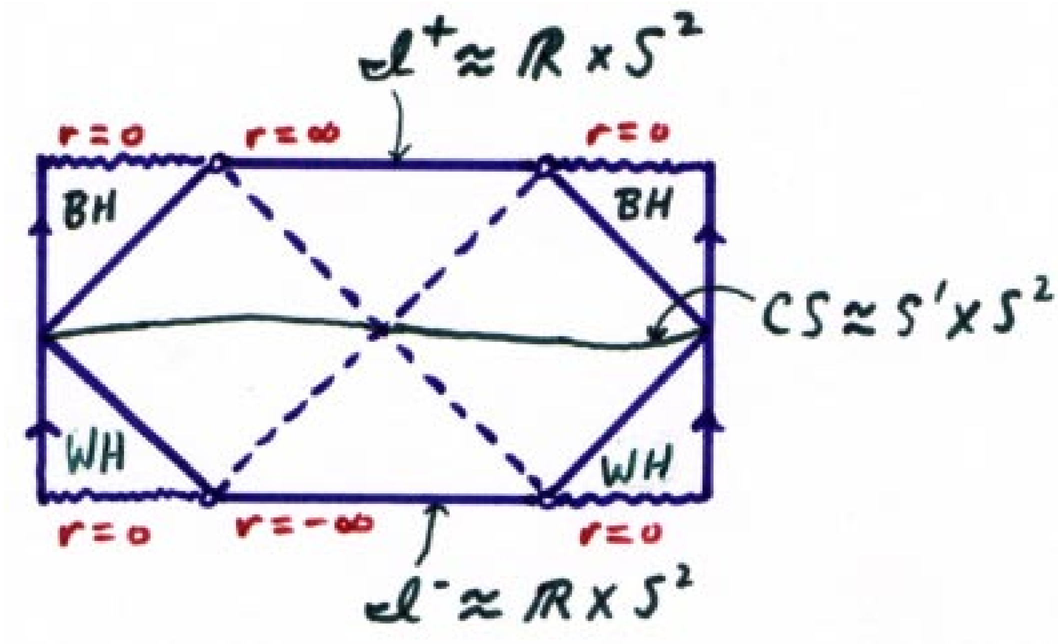
$$M = \mathbb{R} \times S^n, \qquad ds^2 = -dt^2 + \cosh^2 t \, d\Omega^2$$

Ex. Schwarzschild-de Sitter space (dim 4).

$$ds^2 = -(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2)\, dt^2 + (1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2)^{-1} dr^2 + r^2 d\omega^2 \,,$$

where $\Lambda > 0$ (and $9\Lambda m^2 < 1$).

Penrose diagram:



SS-DeS is a spacetime of de Sitter type, but is not asymptotically simple.

<u>*Ex.*</u> FRW spacetime,

$$M = \mathbb{R} \times \Sigma\,, \qquad ds^2 = -dt^2 + a^2(t)d\sigma^2$$

which is a solution to the Einstein equations with perfect fluid source and $\Lambda > 0$.

Starts from a big bang but behaves like de Sitter to the far future.

For such models, $\mathcal{J} = \mathcal{J}^+$, i.e., there is a future conformal infinity, but no past conformal infinity. Shall also refer to such spacetimes as being of de Sitter type.

Asymptotic simplicity can be related to the causal structure of of spacetime.

Prop. *Let M be a spacetime of de Sitter type with future conformal infinity $\mathcal{I}^+$.*

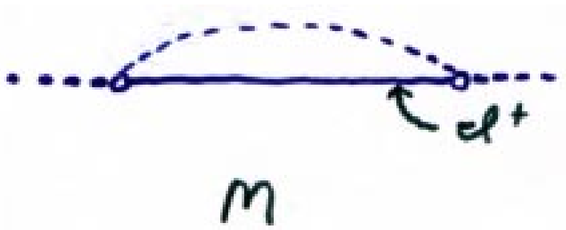
(1) If M is future asymptotically simple then M is globally hyperbolic.

(2) If M is globally hyperbolic and $\mathcal{I}^+$ is compact then M is future asymptotically simple.

In either case, the Cauchy surfaces of M are homeomorphic to $\mathcal{I}^+$.

Comments on proof.

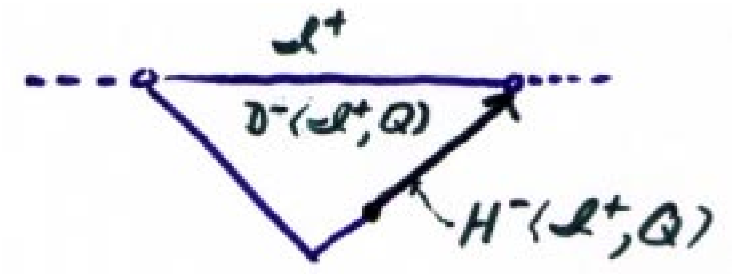
(1): Extend $M \cup \mathcal{I}^+$ a little beyond $\mathcal{I}^+$ to obtain a spacetime without boundary Q such that $\mathcal{I}^+$ is a *future* Cauchy surface in Q:



i.e. such that $D^+(\mathcal{I}^+, Q) = J^+(\mathcal{I}^+, Q) \iff H^+(\mathcal{I}^+, Q) = \emptyset$.

We claim $H^-(\mathcal{I}^+, Q) = \emptyset$, as well, and hence $\mathcal{I}^+$ is a Cauchy surface for Q.

Suppose $H^-(\mathscr{I}^+, Q) \neq \emptyset$:



By asymptotic simplicity, null generators of $H^-(\mathscr{I}^+, Q)$ must meet $\mathscr{I}^+$ $\rightarrow\leftarrow$.

Thus $\mathscr{I}^+$ is Cauchy for Q, and Q is globally hyperbolic. One can then construct a Cauchy surface for Q lying entirely in M. This is easily seen to be a Cauchy surface for M, as well, and hence M is globally hyperbolic.

Finally, since all Cauchy surfaces are homeomorphic, the Cauchy surfaces of M are homeomorphic to $\mathscr{I}^+$.

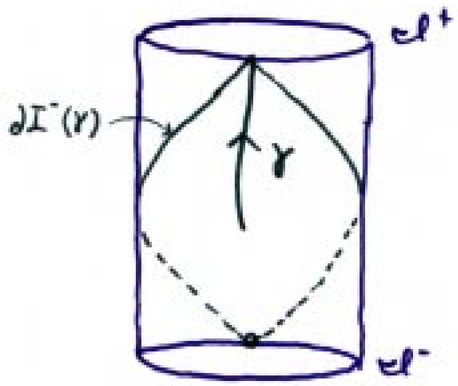
(2): Similar arguments involved. Uses the basic fact:

Prop. If S is a compact achronal hypersurface in a globally hyperbolic spacetime M then S must be a Cauchy surface for M.

Uniqueness results for spacetimes of de Sitter type.

Every null geodesic in de Sitter space is a null line (inextendible achronal null geodesic).

This is related to the fact that the observer horizon $\partial I^-(\gamma)$ of every observer (future inextendible timelike curve) γ is eternal, i.e. extends from $\mathcal{J}^+$ to $\mathcal{J}^-$.



Theorem. *Suppose M^4 is an asymptotically simple spacetime of de Sitter type satisying the vacuum Einstein equation,*

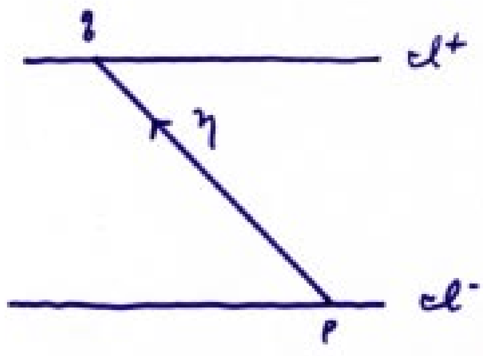
$$\text{Ric} = \lambda g$$

with $\lambda > 0$. If M contains a null line then M is isometric to de Sitter space.

Comment: This can be interpreted in terms of the initial value problem, due to Friedrich's results on the nonlinear stability of asymptotic simplicity, in the case $\Lambda > 0$.

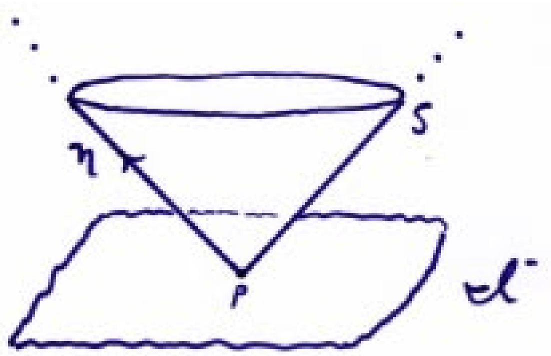
- In general, a small perturbation of the initial data in de Sitter space destroys *all* the null lines, i.e.,
- in the perturbed spacetime, there are no eternal observer horizons.

Proof. The main step is to show M has constant curvature.



η is contained in a smooth totally geodesic null hypersurface S. Focus on situation near p:

$$S = \partial I^+(\eta, M) = \partial I^+(p, \tilde{M}) \cap M$$



Thus, $N_p = S \cup \{p\}$ is a smooth null cone in $\tilde{M}$. Since the shear σ is a conformal invariant, the null generators of N_p have vanishing shear.

The trace free part of the Ricatti equation then implies,

$$\tilde{C}_{aKbK} = 0 \qquad (\iff \tilde{\psi}_0 = 0)$$

By an argument of Friedrich '86,

$$C^i_{jkl} = 0 \qquad \text{on } D^+(N_p, \tilde{M}) \cap M$$

Argument makes use of the conformal field equations, specifically,

$$\tilde{\nabla}_i d^i_{jkl} = 0, \qquad d^i_{jkl} = \Omega^{-1} C^i_{jkl}$$

Time-dually, C^i_{jkl} vanishes on $D^-(N_q, \tilde{M}) \cap M$, and hence on all of M.

Thus M has constant curvature. It can be further shown that M is geodesically complete and simply conncted, and so M is isometric to de Sitter space.

Comments.

(1) The assumption of asymptotic simplicity cannot be removed, cf., SS-deS space. But it appears it can be substantially weakened.

¿Theorem? *Suppose M is a maximally globally hyperbolic spacetime of de Sitter type satisying the vacuum Einstein equation,*

$$\text{Ric} = \lambda g$$

with $\lambda > 0$. If M contains a null line with end points on $\mathcal{J}$ then M is isometric to de Sitter space.

(2) Analogous result holds for Minkowski space.

Theorem. *Suppose M^4 is an asymptotically simple spacetime satisying the vacuum Einstein equation,*

$$\mathrm{Ric} = 0 .$$

If M contains a null line then M is isometric to Minkowski space.

Remarks:

- Due to Corvino-Chrusćiel-Delay, this result is not vacuous!
- Asymptotic simplicity assumption is not so easily weakened in this case.
- This result should continue to hold in the nonvacuum case for certain fields (matter fields, EM, Yang-Mills).

Results on the topology of spacetimes of de Sitter type.

Q. What are the allowable spatial topologies within the class of asymptotically simple and de Sitter solutions of the Einstein equations?

Theorem (Andersson, G.) *Let M^{n+1}, $n \geq 2$, be a spacetime of de Sitter type with past and future conformal boundaries $\mathcal{J}^{\pm}$. Assume that M is asymptotically simple either to the past or future. Assume further that M obeys the null energy condition.*

Then M is globally hyperbolic, and the Cauchy surfaces for M are compact with finite fundamental group.

Comments.

(1) Thus, in $3+1$, the Cauchy surfaces are homotopy 3-spheres, perhaps with identifications.

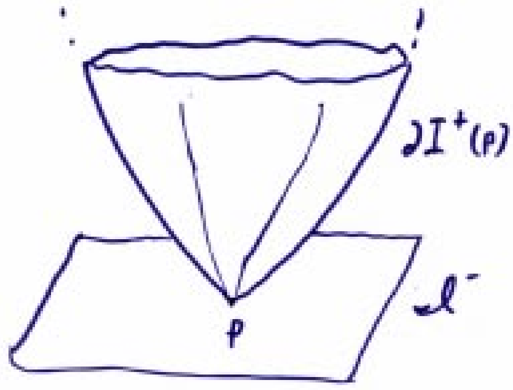
(2) In particular, the Cauchy surfaces cannot have topology $S^2 \times S^1$. Or, put another way if the Cauchy surface topology is $S^2 \times S^1$, then M cannot be asymptotically simple, either to the future or the past; cf., SS-deS.

Proof. We show the Cauchy surfaces of M are compact.

Can extend M a little beyond $\mathcal{J}^{\pm}$ to obtain a spacetime $P \supset \tilde{M}$ such that any Cauchy surface for M is a Cauchy surface for P.

Suffices to show the Cauchy surfaces of P are compact.

Fix $p \in \mathcal{I}^-$, and consider $\partial I^+(p, P)$:

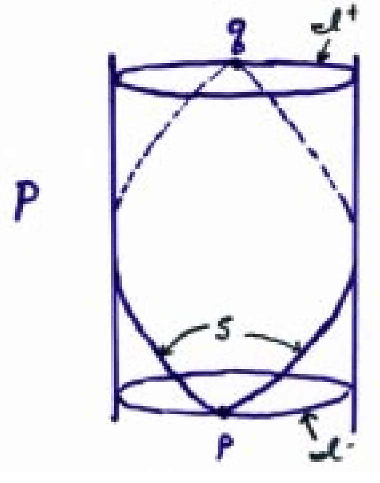


If $\partial I^+(p, P)$ is compact then $\partial I^+(p, P)$ is a compact CS for P and we are done.

If $\partial I^+(p, P)$ is noncompact then can construct null geodesic generator $\gamma \subset \partial I^+(p, P)$ which is future inextendible in P.

By future asymptotic simplicity, γ meets $\mathcal{I}^+$ at q, say. γ_0, the portion of γ from p to q is a null line in M.

By null max prin, γ_0 is contained in a totally geodesic null hypersurface S. By previous arguments, $N = S \cup \{p, q\}$ is a compact achronal hypersurface in P:



Hence, N is a compact Cauchy surface for P. Thus the Cauchy surfaces for M are compact.

We consider a related result which is an application of the *Penrose singularity theorem*,

Theorem (Penrose). *Let M be a globally hyperbolic satisfying the null energy condition. Then the following conditions* cannot *all hold.*

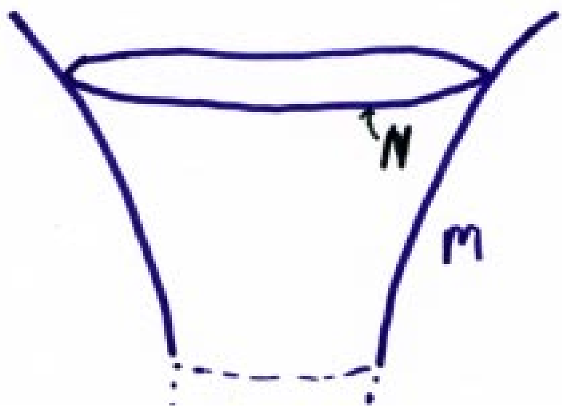
- *The Cauchy surfaces of M are non-compact.*
- *M contains a past-trapped surface.*
- *M is past null geodesically complete.*

Theorem (Andersson, G.) *Suppose M^{n+1}, $2 \leq n \leq 7$, is a globally hyperbolic spacetime of de Sitter type, with future conformal boundary $\mathcal{J}^+$, which is compact and orientable. Suppose further that M obeys the null energy condition.*

If the Cauchy surfaces of M have positive first Betti number, $b_1 > 0$, then M is past null geodesically incomplete.

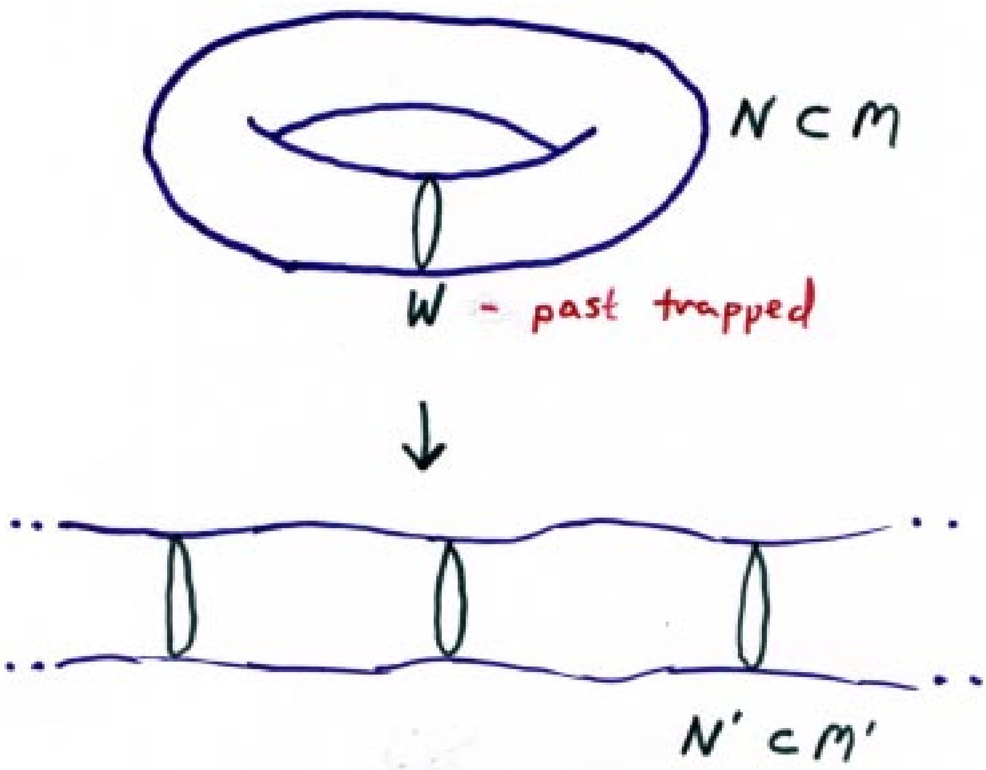
<u>Discussion of proof.</u>

In the far future, can choose a CS N for M, with second fundamental form which is positive definite wrt the *future* pointing normal.



Now, $b_1(N) > 0 \iff H_{n-1}(N, \mathbb{Z}) \neq 0$.

Minimizing area in homology class, obtain a homologically nontrivial smooth compact orientable minimal hypersurface surface $W \subset N$:



The preimage of W in the covering spacetime consists of infinitely many copies of W each past trapped, contained in a noncompact CS. Thus M', and hence M must be past null geodesically complete.

<u>*Comment:*</u> Further results on the topology of spacetimes of de Sitter type may be found in: Andersson and G., hep-th/0202161.