Global Lorentzian Geometry and the Einstein Equations

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Part I

- Causal theory
- Geometry of smooth null hypersurfaces
- Maximum Principle for smooth null hypersurfaces

Part II

- Achronal boundaries
- C^0 null hypersurfaces
- Maximum Principle for C^0 null hypersurfaces
- The null splitting theorem

Part III - Applications

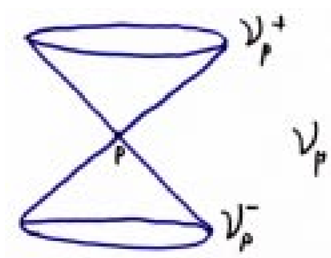
- Uniqueness results for asymptotically de Sitter and asymptotically flat solutions of the vacuum Einstein equations
- Results on the topology of asymptotically de Sitter solutions of the Einstein equations

Elements of Lorentzian Geometry

$$
\begin{aligned}
M^{n+1} &= \text{smooth Lorentzian manifold} \\
&= \text{smooth manifold equipped with metric} \\
&\quad\; g = \langle\ ,\ \rangle \text{ having signature } (- + \cdots +)
\end{aligned}
$$

The null cone $\mathcal{V}_p$ at $p \in M$ is the set,

$$\mathcal{V}_p = \{X \in T_pM; \langle X, X\rangle = g_{ij}X^iX^j = 0\}$$



We always assume M is time orientable , i.e. that the assignment of a past and future cone, $\mathcal{V}_p^-$ and $\mathcal{V}_p^+$, can be made in a continuous manner on M.

spacetime = time oriented Lorentzian manifold

Let,

$$\nabla = \text{Levi-Civita connection}$$

For vector fields $X = X^a$, $Y = Y^b$,

$$\nabla_X Y = X^a \nabla_a Y^b$$

Riemann curvature tensor. For vector fields X, Y, Z,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

The components are determined by,

$$R(\partial_i, \partial_j)\partial_k = R^\ell{}_{kij}\partial_\ell$$

The Ricci tensor and scalar curvature are obtained by tracing,

$$R_{ij} = R^\ell{}_{i\ell j} \quad \text{and} \quad R = g^{ij} R_{ij}$$

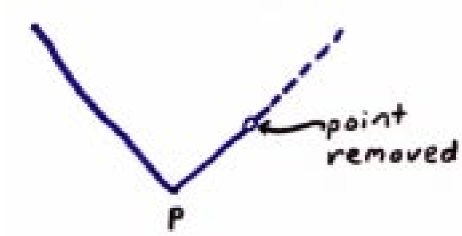
Past and Futures:

Def. For $p \in M$,

$$\begin{aligned} I^+(p) &= \text{timelike future of } p \\ &= \{q \in M : \exists \textit{ future directed} \text{ timelike curve from } p \text{ to } q\} \end{aligned}$$

$$\begin{aligned} J^+(p) &= \text{causal future of } p \\ &= \{q \in M : \exists \textit{ future directed} \text{ causal curve from } p \text{ to } q\} \end{aligned}$$

Note: $I^+(p)$ is always open, but $J^+(p)$ need not be closed.

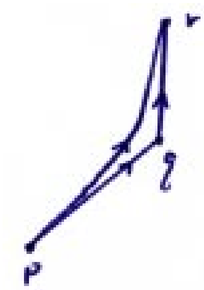


Def. For $A \subset M$,

$$I^+(A) = \{q \in M : \exists \textit{ future directed } \text{timelike curve from some } p \in A \text{ to } q\}$$
$$= \cup_{p \in A} I^+(p) \qquad \text{(always open)}$$

$$J^+(A) = \{q \in M : \exists \textit{ future directed } \text{causal curve from some } p \in A \text{ to } q\}$$
$$= \cup_{p \in A} J^+(p)$$

Prop. $q \in J^+(p)$ and $r \in I^+(q) \Rightarrow r \in I^+(p)$, etc.



Prop. *If $q \in J^+(p) \setminus I^+(p)$ then any causal curve γ from p to q must be a null geodesic.*

Note: $I^-(p)$, $J^-(p)$, $I^-(A)$, $J^-(A)$ defined time-dually.

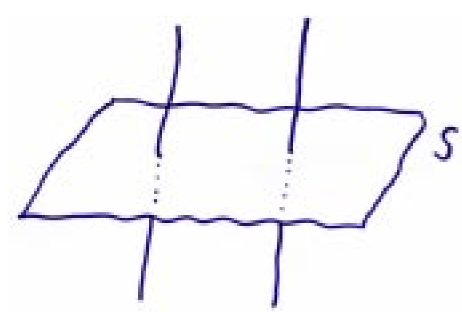
Global hyperbolicity:

Def. *Strong Causality* holds at $p \in M$ provided there are arbitrarily small neighborhoods U of p such that any causal curve γ which starts in, and leaves, U never returns to U.

Def. M is globally hyperbolic provided

- M is strongly causal
- The sets $J^+(p) \cap J^-(q)$ are compact $\forall p, q \in M$

Def. A Cauchy surface for M is an achronal C^0 hypersurface S in M which is met by every inextendible causal curve in M.



Comment: Equivalently, an achronal hypersurface S is Cauchy provided $D(S) = M \iff H(S) = \emptyset$

Prop. M is globally hyperbolic iff M admits a Cauchy surface.

Prop. If S is a Cauchy surface for M then M is homeomorphic to $\mathbb{R} \times S$.

(Moreover the homeomorphism can be arranged so that $\{t\} \times S$ is Cauchy $\forall t$.)

Prop. If S is a compact achronal hypersurface in a globally hyperbolic spacetime M then S must be a Cauchy surface for M.

Prop. If M is globally hyperbolic then

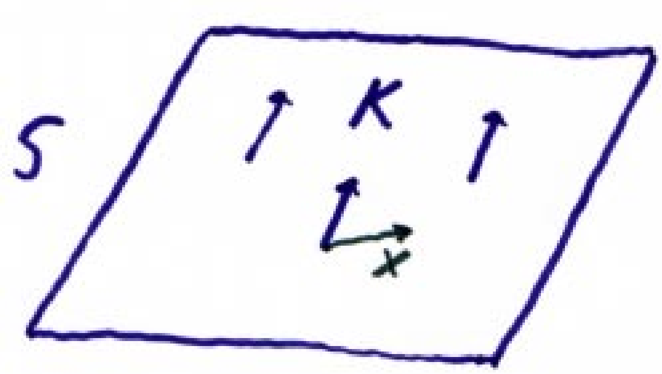
- $J^{\pm}(A)$ are closed $\forall A \subset M$ compact.
- $J^{+}(A) \cap J^{-}(B)$ is compact $\forall A, B \subset M$ compact.

Geometry of Null Hypersurfaces

Def. A smooth null hypersurface in (M, g) is a smooth co-dim one submanifold S of M, such that the pullback of g to S is degenerate.

Such an S admits a smooth future directed null tangent vector field K such that

$$[K_p]^\perp = T_pS \qquad \forall p \in S$$



<u>*Note:*</u>

- Every vector X tangent to S, and not a multiple of K, is spacelike.
- K is unique up to a positive pointwise scale factor.

Prop. *The integral curves of K, when suitably parameterized, are null geodesics (and are called the null generators of S).*

Proof: Suffices to show:

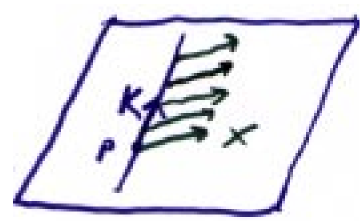
$$\nabla_K K = \lambda K$$

This follows by showing at each $p \in S$,

$$\nabla_K K \perp T_p S\,, \quad \text{i.e.,} \quad \langle \nabla_K K, X\rangle = 0 \quad \forall X \in T_p S$$

Extend $X \in T_p S$ by making it invariant under the flow generated by K,

$$[K, X] = \nabla_K X - \nabla_X K = 0$$



X remains tangent to S, so along the flow line through p,

$$\langle K, X\rangle = 0$$

Differentiating,

$$K\langle K, X\rangle = \langle \nabla_K K, X\rangle + \langle K, \nabla_K X\rangle = 0$$

$$\langle \nabla_K K, X\rangle = -\langle K, \nabla_X K\rangle = -\frac{1}{2} X\langle K, K\rangle = 0.$$

QED

Null Weingarten Map/Null 2nd Fundamental Form.

One works mod K: For $X, Y \in T_pS$,

$$X = Y \text{ mod } K \iff X - Y = \lambda K$$

Let $\overline{X}$ denote equivalence class of $X \in T_pS$ and let,

$$T_pS/K = \{\overline{X} : X \in T_pS\}$$

Then,

$$TS/K = \cup_{p\in S} T_pS/K$$

is a rank $n-1$ vector bundle over S ($n = \dim S$).

Positive definite metric on TS/K:

$$\begin{aligned} h : T_pS/K \times T_pS/K &\to \mathbb{R} \\ h(\overline{X}, \overline{Y}) &= \langle X, Y\rangle \end{aligned}$$

Well-defined: $X' = X$ mod K, $Y' = Y$ mod $K \Rightarrow$

$$\begin{aligned} \langle X', Y'\rangle &= \langle X + \alpha K, Y + \beta K\rangle \\ &= \langle X, Y\rangle + \beta\langle X, K\rangle + \alpha\langle K, Y\rangle + \alpha\beta\langle K, K\rangle \\ &= \langle X, Y\rangle \end{aligned}$$

Weingarten Map:

$$b : T_pS/K \rightarrow T_pS/K$$
$$b(\overline{X}) = \overline{\nabla_X K}$$

Well-defined: $X' = X$ mod $K \Rightarrow$

$$\begin{aligned} \nabla_{X'}K &= \nabla_{X+\alpha K}K \\ &= \nabla_X K + \alpha \nabla_K K = \nabla_X K + \alpha\lambda K \\ &= \nabla_X K \quad \text{mod } K \end{aligned}$$

Second Fundamental Form:

$$\begin{aligned} B : T_pS/K \times T_pS/K &\rightarrow \mathbb{R} \\ B(\overline{X}, \overline{Y}) &= h(b(\overline{X}), \overline{Y}) \\ &= \langle \nabla_X K, Y \rangle \end{aligned}$$

Prop. *B is symmetric, $B(\overline{X}, \overline{Y}) = B(\overline{Y}, \overline{X}), \forall \overline{X}, \overline{Y} \in T_pS/K$, and hence b is self-adjoint.*

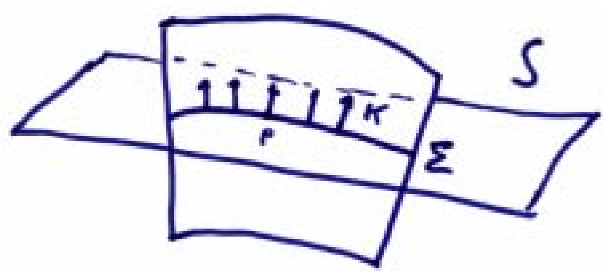
Proof. Extend X, Y to vector fields tangent to S near p. Using $X\langle K, Y\rangle = 0$ and $Y\langle K, X\rangle = 0$,

$$\begin{aligned} B(\overline{X}, \overline{Y}) &= \langle \nabla_X K, Y \rangle = -\langle K, \nabla_X Y \rangle \\ &= -\langle K, \nabla_Y X \rangle + \langle K, [X, Y] \rangle \\ &= \langle \nabla_Y K, X \rangle = B(\overline{Y}, \overline{X}) \end{aligned}$$

Null mean curvature (expansion scalar):

$$\begin{aligned}\theta &= \operatorname{tr} b \\ &= \operatorname{div} K \quad \text{(essentially)}\end{aligned}$$

Let Σ be the intersection of S with a hypersurface in M which is transverse to K near $p \in S$; Σ will be an $n-1$ dimensional spacelike submanifold of M.

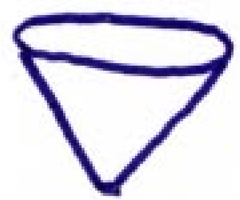


Let $\{e_1, e_2, \cdots, e_{n-1}\}$ be an orthonormal basis for $T_p\Sigma$ in the induced metric. Then $\{\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_{n-1}\}$ is an orthonormal basis for T_pS/K.

Hence,

$$\begin{aligned}\theta = \operatorname{tr} b &= \sum_{i=1}^{n-1} h(b(\overline{e}_i), \overline{e}_i) \\ &= \sum_{i=1}^{n-1} \langle \nabla_{e_i} K, e_i \rangle \\ &= \operatorname{div}_\Sigma K \quad \text{at } p\end{aligned}$$

Thus, θ measures the expansion of the null generators of S towards the future.



$\theta > 0$

$\theta < 0$

Effect of scaling K:

Prop. $\widetilde{K} = fK \Rightarrow b_{\widetilde{K}} = f\, b_K$, and hence $\theta_{\widetilde{K}} = f\, \theta_K$

Proof:

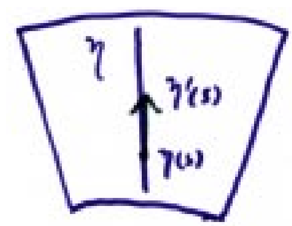
$$\nabla_X \widetilde{K} = \nabla_X(fK) = X(f)K + f\nabla_X K = f\nabla_X K \quad \text{mod} K$$

Note: In particular, the Weingarten map $b = b_K$ at a point $p \in S$ is uniquely determined by the value of K at p.

Comparison theory.

Let $\eta : I \to M$, $s \to \eta(s)$, be an affinely parameterized null geodesic generator of S, and let

$$b(s) = b_{\eta'(s)}$$



be the null Weingarten map at $\eta(s)$ wrt the null tangent vector $\eta'(s)$.

The family of Weingarten maps $b = b(s)$ along η obeys the Ricatti equation,

$$\boxed{b' + b^2 + R = 0, \qquad ' = \nabla_{\eta'}}$$

where, by def.,

$$b'(\overline{X}) = b(\overline{X})' - b(\overline{X}'), \quad (\text{and } (\overline{Y})' = \overline{Y'})$$

$$R(\overline{X}) = \overline{R(X,\eta')\eta'}\,.$$

Proof:

Fix $p = \eta(s_0)$ on η, and scale K so that in a neighborhood of p,

(i) K is geodesic, $\nabla_K K = 0$.

(ii) $K = \eta'$ along η.

Extend $X \in T_pS$ near p by making it invariant under the flow generated by K,

$$[K, X] = \nabla_K X - \nabla_X K = 0\,.$$

Then,

$$R(X, K)K = \nabla_X \nabla_K K - \nabla_K \nabla_X K - \nabla_{[X,K]} K = -\nabla_K \nabla_K X\,,$$

i.e., along η, X satisfies,

$$X'' = -R(X, \eta')\eta'\,.$$

Thus,

$$\begin{aligned}
b'(\overline{X}) &= \overline{\nabla_X K}' - b(\overline{\nabla_K X}) = \overline{\nabla_K X}' - b(\overline{\nabla_X K}) \\
&= \overline{X''} - b(b(\overline{X})) = -\overline{R(X, \eta')\eta'} - b^2(\overline{X}) \\
&= -R(\overline{X}) - b^2(\overline{X})
\end{aligned}$$

QED

By tracing, we obtain along η that $\theta = \theta(s)$ obeys,

$$\frac{d\theta}{ds} \;=\; -\mathrm{Ric}\,(\eta', \eta') - \mathrm{tr}\, b^2\,,$$

or,

$$\boxed{\frac{d\theta}{ds} = -\mathrm{Ric}\,(\eta', \eta') - \sigma^2 - \frac{1}{n-1}\theta^2}$$

Raychaudhuri's equation

where σ is the *shear* scalar, $\sigma^2 = \mathrm{tr}\,\widehat{b}^2$, $\widehat{b} = b - \frac{1}{n-1}\theta \cdot \mathrm{id}$.

Prop. *Let S be a smooth null hypersurface in a spacetime M which obeys the null energy condition,*

$$\mathrm{Ric}\,(X, X) \geq 0 \quad \forall \text{ null vectors } X\,.$$

Then, if the null generators of S are future geodesically complete, S has nonnegative null mean curvature, $\theta \geq 0$.

Proof. Suppose $\theta < 0$ at $p \in S$. Let $s \to \eta(s)$ be the null generator of S passing through $p = \eta(0)$.

Let $b(s) = b_{\eta'(s)}$, and take $\theta = \mathrm{tr}\; b$. By invariance of sign under scaling, $\theta(0) < 0$.

By Raychaudhuri's equation and the NEC,

$$\frac{d\theta}{ds} \le -\frac{1}{n-1}\theta^2\,,$$

and hence $\theta < 0$ for $s > 0$. Dividing through by θ^2 gives,

$$\frac{d}{ds}\left(\frac{1}{\theta}\right) \ge \frac{1}{n-1}\,,$$

which implies $1/\theta \to 0$, i.e., $\theta \to -\infty$ in finite affine parameter time, $\to\leftarrow$.

Re: Hawking area theorem; cf., Chruściel, Delay, G. Howard (2001).

Totally geodesic null hypersurfaces.

By def., a smooth null hypersurface S is *totally geodesic* iff $B \equiv 0$ (or, equivalently, iff $\theta = \sigma = 0$).

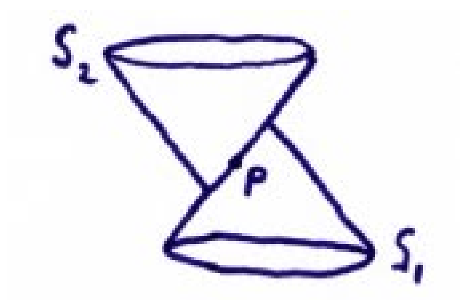
Prop. *A null hypersurfaces S is totally geodesic iff geodesics starting tangent to S remain in S.*

Ex. *Null hyperplanes in Minkowski space, the event horizon in Schwarzschild are totally geodesic.*

Maximum Principle for Smooth Null hypersurfaces.

Theorem. *Suppose*

- *S_1 and S_2 are smooth null hypersurfaces in M.*
- *S_1 , S_2 meet at $p \in M$, with S_2 to the future side of S_1 near p.*



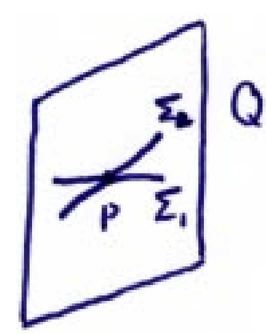
- $\theta_2 \leq 0 \leq \theta_1$.

Then S_1 and S_2 coincide near p, and $\theta_1 = \theta_2 = 0$.

Proof:

S_1 and S_2 have a common null direction at p. Let Q be a timelike hypersurface in M passing through p and transverse to this direction. Consider the intersections,

$$\Sigma_1 = S_1 \cap Q\,, \qquad \Sigma_2 = S_2 \cap Q$$



Σ_1 and Σ_2 are *spacelike* hypersurfaces in Q, with Σ_2 to the future of Σ_1 near p.

Express Σ_1 and Σ_2 as graphs over a fixed hypersurface V in Q,

$$\Sigma_1 = \text{ graph}\,(u_1)\,, \qquad \Sigma_2 = \text{ graph}\,(u_2)$$

Let,

$$\theta(u_i) = \theta_i|_{\Sigma_i = \text{ graph}\,(u_i)}\,, \quad i = 1, 2$$

By a computation,

$$\theta(u_i) = H(u_i) + \text{ l.o.t.}$$

where $H =$ mean curvature operator on spacelike graphs over V in Q. Thus θ is a second order quasi-linear elliptic operator.

We have:

- $u_1 \le u_2$, and $u_1(p) = u_2(p)$.
- $\theta(u_2) \le 0 \le \theta(u_1)$.

By the strong maximum principle, $u_1 = u_2$.

Thus, S_1 and S_2 agree near p in Q. Now, vary Q to get agreement on a neighborhood.