

The Conformal Method

Goal:

Parametrize
&
Construct

Solutions $(\Sigma^3; \gamma, K, \psi, \pi)$
of the Constraint Eqs:

$$R - K K + (G \cdot K)^2 = \rho$$

$$\nabla \cdot K - \nabla (L \cdot K) = J$$

underdetermined
system

Idea:

Split (γ, K, ψ, π) into

Free Data

&

Constrained Data

\Rightarrow Determined Elliptic System

Payoff:

- Parametrization
for various regularities
- Numerical Construction Scheme

Most Used

• Tool for Gluing

- * Add wormholes to your spacetime
- * Add Black holes to your spacetime
- * Verify: There exists Cauchy data on every Σ

• Build In Physics

Limitations:

• Tool for Gluing

Determined vs Under Determined System

• Build In Physics

3 Carrying Out the Conformal Method

Note: Developers

Lichnerowicz

Choquet-Bruhat

York

O'Murchadha

I

First Seen

Racine (1970s)

Note: Stick to

Vacuum Einstein

But...

Split of Fields

Free Data

λ_{ab} - conformal Riem metric

O^{cd} - div-free & trace-free sym tensor field

$$\nabla \cdot O = 0$$

$$\lambda_{ab} O^{ab} = 0$$

\mathcal{Q} - function

Determined Data

W^m - vector field

Φ - pos def function

Determining Equations

$$\nabla_a (LW)^a_b = \phi^6 \nabla_b \tau$$

$$\nabla^2 \phi = R \phi - (\sigma + LW)^2 \phi^{-7} + \tau^2 \phi^5$$

$$\uparrow$$
$$LW = \nabla_a W_b + \nabla_b W_a - \frac{2}{3} \lambda_{ab} \nabla \cdot W$$

- Given $\{\Sigma^3; \lambda, \sigma, \tau\}$
find $\{W, \phi\}$ solving these
- Determined system
 - * 4 eqns
 - * 4 unknowns

Solution of Constraints

$$\sigma = \phi^4 \lambda$$

$$k = \phi^{-10} (\sigma + LW) + \frac{2}{3} \phi^{-4} \lambda \tau$$

is soln

• Does it always work

No!

Pick $\Sigma^3 = S^3$

λ with $R \geq 0$

$\sigma = 0$

$\tau = 1$

$$\rightarrow \nabla(LW) = 0$$

$$\rightarrow LW = 0$$

$$\rightarrow \nabla^2 \phi = R\phi + \phi^5$$

> 0

\rightarrow No Solution

Big Question:

When does it work?

Classifying "When"

- Manifold & Asymptotic Conditions

- Closed
- Asymptotically Euclidean
- Asymptotically Hyperbolic
- Compact with Boundary

- Regularity

- Smooth
- Sobolev & Holder Spaces

$$H_k, H_k^p, H_{k,s}^p, C^{k,\alpha}$$

* Optimized Regularity

- Specifics of Conformal Data

- Yamabe Class of λ_0
- CMC or Non CMC
- $G \geq 0$ $G \neq 0$
- Other Fields