Asymptotically Flat Initial Data with Prescribed Regularity at Infinity

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Cargse July 29 - August 10, 2002

Initial data set for the Einstein equation

An initial data set for the Einstein vacuum equations is a triplet $(\tilde{S}, \tilde{h}_{ab}, \tilde{\Psi}_{ab})$ where

– $\tilde{S}$: 3-dimensional manifold.

– $\tilde{h}_{ab}$: Riemannian metric.

– $\tilde{\Psi}_{ab}$: symmetric 2-tensor.

Which satisfy the constraint equations on $\tilde{S}$

$$\tilde{D}^a\tilde{\Psi}_{ab} - \tilde{D}_a\tilde{\Psi} = 0,$$

$$\tilde{R} + \tilde{\Psi}^2 - \tilde{\Psi}_{ab}\tilde{\Psi}^{ab} = 0,$$

where

– $\tilde{D}_a$: covariant derivative of $\tilde{h}_{ab}$.

– $\tilde{R}$: Ricci scalar of $\tilde{h}_{ab}$.

– $\tilde{\Psi} = \tilde{h}^{ab}\tilde{\Psi}_{ab}$.

Asymptotically flat initial data set

We say that an initial data set is asymptotically flat if the complement of a compact set in $\tilde{S}$ is diffeomorphic to the complement of a ball in $\mathbb{R}^3$, and there exists a coordinate system $\tilde{x}^j$ in a neighborhood of infinity such that, in this coordinates,

$$\tilde{h}_{ij} = (1 + \frac{2m}{\tilde{r}})\delta_{ij} + O(\tilde{r}^{-2}),$$

$$\tilde{\Psi}_{ij} = O(\tilde{r}^{-2}),$$

as

$$\tilde{r} = (\sum_{j=1}^{3} (\tilde{x}^j)^2)^{1/2} \to \infty.$$

Where the constant m is the mass of the data, and δ_{ij} is the flat metric.

The Problem we want to solve

We want to consider the problem of the existence of a class of asymptotically flat initial data for which the higher order terms of $\tilde{h}_{ab}$ and $\tilde{\Psi}_{ab}$ have an asymptotic expansion in powers of $\tilde{r}$ of the form

$$\tilde{h}_{ij} \sim (1 + \frac{2m}{\tilde{r}})\delta_{ij} + \sum_{k\geq 2} \frac{\tilde{h}^k_{ij}}{\tilde{r}^k},$$

$$\tilde{\Psi}_{ij} \sim \sum_{k\geq 2} \frac{\tilde{\Psi}^k_{ij}}{\tilde{r}^k},$$

where $\tilde{h}^k_{ij}$ and $\tilde{\Psi}^k_{ij}$ are smooth functions of $\tilde{x}^j/r$.

Maximal initial data

If we assume that

$$\tilde{\Psi} = 0,$$

then the constraint equations reduce to

$$\tilde{D}^a \tilde{\Psi}_{ab} = 0,$$

$$\tilde{R} - \tilde{\Psi}_{ab} \tilde{\Psi}^{ab} = 0.$$

After a conformal rescaling

$$\tilde{h}_{ab} = \theta^4 h_{ab}, \quad \tilde{\Psi}_{ab} = \theta^{-2} \Psi_{ab},$$

we have

$$D^a \Psi_{ab} = 0 \text{ on } \tilde{S}, \tag{1}$$

$$(D_b D^b - \frac{1}{8} R)\theta = -\frac{1}{8} \Psi_{ab} \Psi^{ab} \theta^{-7} \text{ on } \tilde{S}. \tag{2}$$

Conformal compactification of the initial data

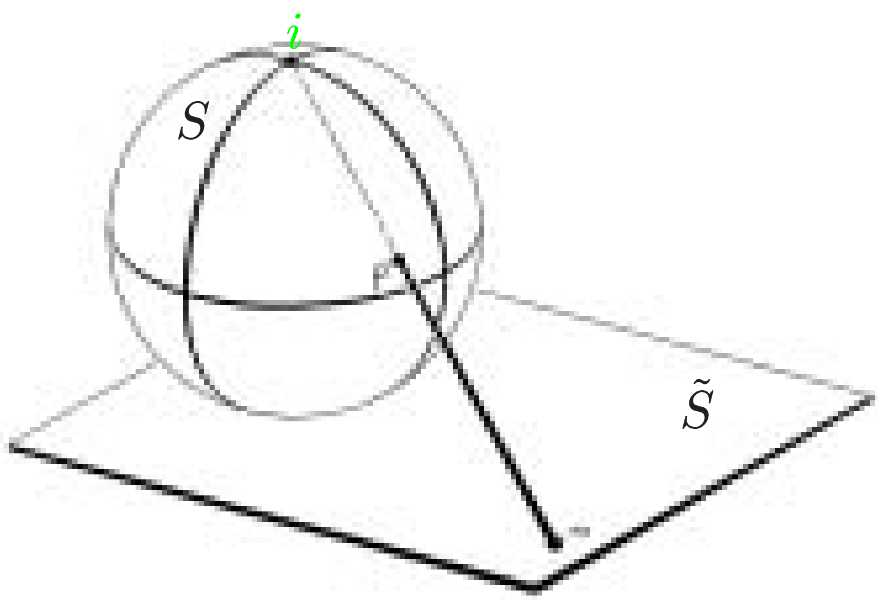
Let S be the compact manifold defined by $S = \tilde{S} \cup \{i\}$, and define coordinates near the point i by the inversion

$$x^i = \frac{\tilde{x}^i}{\tilde{r}^2}, \qquad r = (\sum_{j=1}^{3} (x^j)^2)^{1/2} = \frac{1}{\tilde{r}}.$$

Then the asymptotically flat condition for the initial data are

$$\Psi_{ab} = O(r^{-4}) \quad \text{as} \quad r \to 0, \tag{3}$$

$$\lim_{r \to 0} r\theta = 1. \tag{4}$$



Our result

B_a : small ball center at i

Definition 1

$$E^\infty(B_a) = \{f = f_1 + r f_2 \,:\, f_1, f_2 \in C^\infty(B_a)\}$$

Theorem 1 *Let h_{ab} be a smooth metric on S with positive Ricci scalar R. Assume that Ψ_{ab} is smooth in $\tilde{S}$ and satisfies on B_a*

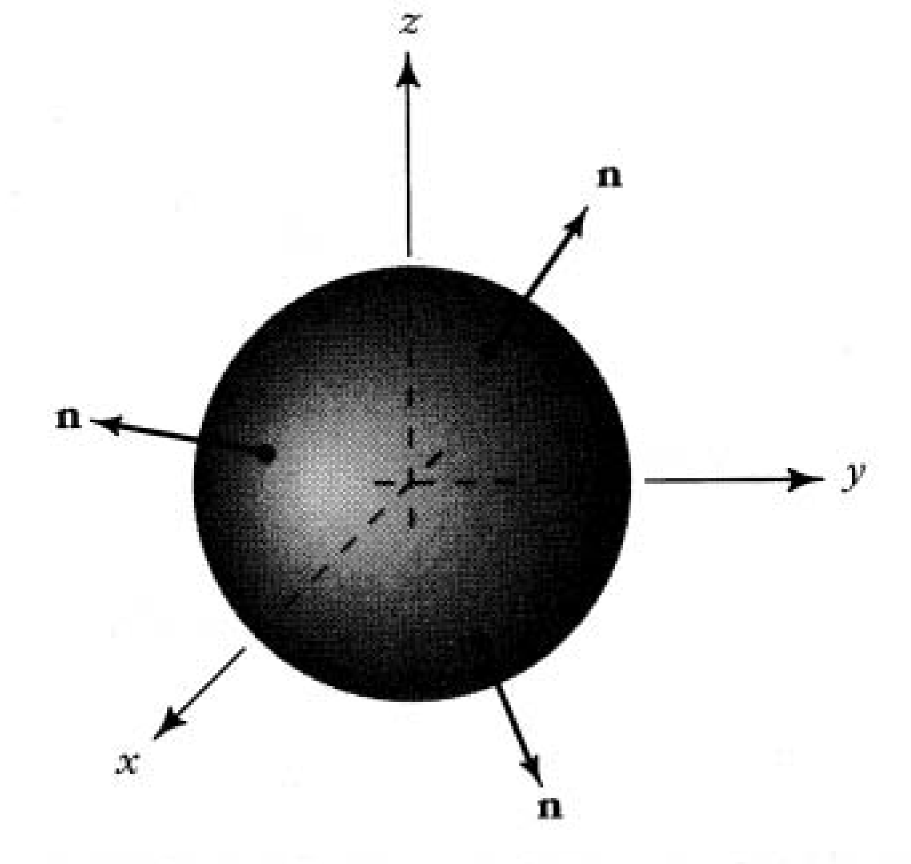
$$r^8 \Psi_{ab} \Psi^{ab} \in E^\infty(B_a). \qquad (5)$$

Then there exists on $\tilde{S}$ a unique solution θ of equation (2), which is positive, satisfies (4), and has in B_a the form

$$\theta = \frac{\widehat{\theta}}{r}, \quad \widehat{\theta} \in E^\infty(B_a), \quad \widehat{\theta}(i) = 1.$$

General solutions of $\partial_a \Psi^{ab} = 0$

$(n^a, m^a, \bar{m}^a)$: orthonormal, complex, tetrad.



$$\xi = \frac{1}{2} r^3 \Psi_{ab} n^a n^b, \quad \eta_1 = \sqrt{2} r^3 \Psi_{ab} n^a m^b,$$

$$\mu_2 = r^3 \Psi_{ab} m^a m^b.$$

λ arbitrary complex function and $\lambda_2 = \eth^2 \lambda$.

Then, the general solution of $\partial_a \Psi^{ab} = 0$ is given by

$$\xi = \bar{\eth}^2 \lambda_2^R + A + r\, Q + \frac{1}{r}\, P,$$

$$\eta_1 = -2\, r\, \partial_r\, \bar{\eth} \lambda_2^R + \bar{\eth} \lambda_2^I + r\, \eth Q - \frac{1}{r}\, \eth P + i\, \eth J,$$

$$\mu_2 = 2\, r\, \partial_r (r\, \partial_r\, \lambda_2^R) - 2\, \lambda_2^R + \eth \bar{\eth} \lambda_2^R - r\, \partial_r\, \lambda_2^I.$$

where

$$P = \frac{3}{2} P^a n_a, \quad Q = \frac{3}{2} Q^a n_a, \quad J = 3 J^a n_a,$$

and A, P^a, Q^a, J^a are arbitrary constants.

$$
\begin{aligned}
\Psi^{ab}_P &= \frac{3}{2r^4}\left(-P^a n^b - P^b n^a - (\delta^{ab} - 5n^a n^b)\, P^c n_c\right),\\
\Psi^{ab}_J &= \frac{3}{r^3}(n^a \epsilon^{bcd} J_c n_d + n^b \epsilon^{acd} J_c n_d),\\
\Psi^{ab}_A &= \frac{A}{r^3}\,(3n^a n^b - \delta^{ab}),\\
\Psi^{ab}_Q &= \frac{3}{2r^2}\left(Q^a n^b + Q^b n^a - (\delta^{ab} - n^a n^b)\, Q^c n_c\right).
\end{aligned}
$$

λ gives the "quadrupolar" of Ψ^{ab}, it doesn't contribute to the linear or angular momentum of the data.

Theorem 2 *If* $r\lambda \in E^\infty(B_a)$ *and* $P^a = 0$, *then* $r^8 \Psi_{ab}\Psi^{ab} \in E^\infty(B_a)$.