0.0.1 Constraints. LCBY equations

Thin sanwich formulation. Given:

riemannian manifold (M, γ), scalar τ.

Traceless tensor u^{ij}, scalar N.

Sources, fields and matter: 4 scalars $\rho_I \geq 0$, 2 vectors J_1, J_2.

Unknowns: scalar φ, vector β

Equations

$$\mathcal{H} \equiv \Delta_\gamma \varphi - f(., \beta, \varphi) = 0,$$

$$f(., \beta, \varphi) \equiv \tilde{r}\varphi - a\varphi^{-P} - q_1\varphi^{-P_1} + (b - q_2)\varphi^{Q},$$

P=$\frac{3n-2}{n-2}, P_1 = \frac{n}{n-2}, Q = \frac{n+2}{n-2}, k_n = \frac{n}{4n-1}$

$$\tilde{r} \equiv r - q_3, r \equiv k_n R(\gamma), a \equiv k_n A.A + q_0 \geq 0$$

$$b \equiv \frac{n-2}{4n}\tau^2 \geq 0, q_I \equiv 2k_n\rho_I \geq 0.$$

$$A^{ij} \equiv (2N)^{-1}\{(\mathcal{L}_{\gamma,conf}\beta)^{ij} - u^{ij}\}$$

$$\mathcal{M}^i \equiv D_j A^{ij} - \{\frac{n-1}{n}\varphi^{2n/(n-2)}\partial^i\tau + \varphi^{2(n+2)/(n-2)}J_2 + J_1\} = 0.$$

Note: ψ scalar field (source)

$$2\rho_0 \equiv N^{-2}|\partial_0\psi|^2, 2\rho_3 \equiv \gamma^{ij}\partial_i\psi\partial_j\psi$$

0.0.2 Momentum constraint

Suppose momentarily φ is known if $\partial\tau \not\equiv 0$ or $J_2 \not\equiv 0$. Linear elliptic for β .

Theorem. (as. euc. case)

$(M, \gamma) : W^p_{\sigma,\rho}$ as. euc., $N > 0$, $N - 1 \in W^p_{\sigma,\rho}$, $\sigma > \frac{n}{p} + 1$, $\rho > -\frac{n}{p}$, $p > \frac{n}{2}$.

u, $\tau \in W^p_{s_0+1,\delta+1}$, $J_1, J_2 \in W^p_{s_0,\delta+2}$, $s_0 \geq 0$, $\varphi > 0$, $(1 - \varphi) \in W^p_{s_0+2,\delta}$.

Then the momentum constraint has one and only one solution $\beta \in W^p_{s+2,\delta}$ with $0 \leq s \leq Max(s_0, \sigma - 2)$, $-\frac{n}{p} < \delta < n - 2 - \frac{n}{p}$

Case of compact M, same result without the weights, if (M, γ) has no conformal Killing vector, or, when it has one, if the given data and sources are invariant under it.

0.0.3 Theorem. (M compact)

The (L) equation in vacuum on (M,γ), $\gamma \in W^p_\sigma$, $\sigma \geq \frac{n}{p}+1$, a, $b \in L^p \cap L^\infty$, $p > \frac{n}{2}$, admits a solution $\varphi > 0$, $\varphi \in W^p_2$ if:

1. (M,γ) is in the positive Yamabe class, $r(\gamma) = 1$ and $a \not\equiv 0$ on M.

2. (M,γ) is in the zero Yamabe class, $r(\gamma)$= 0, $\tau^2 \not\equiv 0$ and $a \not\equiv 0$ on M.

3.(M,γ) is in the negative Yamabe class, r(γ) $= -1$ and Inf $\tau^2 > 0$ on a sufficiently large subset of M.

Proof: super and subsolution method. Local conformal transformations when necessary.

Case of coupling with a scalar field:

$\tilde{a} = a + q_0$, $\tilde{r} = r - q_3$, $\tilde{b} = b - m\psi^2$

0.0.4 Definition

(M,γ) is (p,σ,ρ) as. euc. if $\gamma - e \in W^p_{\sigma,\rho}$, $\sigma > \frac{n}{p}+1$, $\rho > -\frac{n}{p}$, then $\gamma - e \in C^1_\alpha$, $\alpha > 0$.

0.0.5 Theorem.

Let (M,γ) be (p,σ,ρ) as. euc. i.e. $\gamma - e \in W^p_{\sigma,\rho}$, $\sigma > \frac{n}{p}+1, \rho > -\frac{n}{p}$.

The operator $\Delta_\gamma - k$, $k \in W^p_{s_0,\delta_0+2}$, $s_0 \geq 0$, $\delta_0 > -\frac{n}{p}$, is an isomorphism from $W^p_{s+2,\delta}$ onto $W^p_{s,\delta+2}$, with $0 \leq s \leq Max(\sigma-1,s_0)$, if

$$p > \frac{n}{2}, \quad -\frac{n}{p} < \delta < -\frac{n}{p} + n - 2.$$

and if

$$\int_M \{|\partial f|^2 + kf^2\}\mu_\gamma > 0 \quad \text{for all} \quad f \in \mathcal{D}, \;\; f \not\equiv 0 \; .$$

Proof. $\Delta_\gamma - k$, $k \in W^p_{0,\delta_0+2}$, $\delta_0 > -\frac{n}{p}$, is injective on $W^p_{2,\delta}$ if $p > \frac{n}{2}$, $\delta > -\frac{n}{p}$, because if $u \in \ker(\Delta_\gamma - k) \cap W^p_{2,\delta}$, $\delta > -\frac{n}{p}$, then $u \in W^p_{2,\delta}$ for any $\delta < -2 + n - \frac{n}{p}$, hence for some $\delta > -1 + \frac{n}{2} - \frac{n}{p}$.

Integration after product by u and Holder inequality complete the proof.

0.0.6 Maximum principle.

If $c-\theta\in W^p_{2,\delta}$, $p>\frac{n}{2}$, $\delta>-\frac{n}{p}$, with c a number ≥ 0 and if θ satisfies the equation

$$\Delta_\gamma\theta-k\theta=-f$$

with $\gamma-e\in W^p_{\sigma,\rho},\sigma>\frac{n}{p}+1$, $\rho>-\frac{n}{p}$, $k\in W^p_{0,\delta+2},k\geq 0$, and $f\geq 0$, then $\theta\geq 0$ on M.

Proof. The lemma holds by the classical maximum principle when $c-\theta\in W^p_{s+2,\delta}$ if $s>\frac{n}{p}$, $\delta>-\frac{n}{p}$, and k is bounded, since then $\theta\in C^2$ and tends to c at infinity. Approximate f and k in $W^p_{0,\delta+2}$ by sequences $f_n,k_n\in W^p_{s,\delta+2}$, $s>\frac{n}{p}$, $f_n\geq 0$, $k_n\geq 0$ and use the isomorphism theorem.

Theorem (Lichnerowicz equation, scaled sources, n=3).

Let (M,γ) be a (p,σ,ρ) asymptotically euclidean 3- manifold (M,γ), $\sigma > \frac{3}{p}+1, \rho > -\frac{3}{p}$, $p > \frac{3}{2}$ in the positive Yamabe class, conformally transformed so that $r(\gamma)=0$. The equation

$$\Delta_\gamma \varphi + a\varphi^{-7} + q\varphi^{-3} - b\varphi^5 = 0,$$

$$a, q, b \in W^p_{0,\delta+2}, \delta > -\frac{3}{p},\ a, q, b \geq 0,$$

has one and only one solution, $\varphi = 1+u$, $u \in W^p_{2,\delta}$, $\varphi > 0 >$, if $\delta < 1 - \frac{3}{p}$. It can be obtained by iteration.

If moreover $a, q, b \in W^p_{s_0,\delta+2}$, $s_0 \geq 0$, then $u \in W^p_{s+2,\delta}$, $s = Min(s_0, \sigma - 1)$.

0.0.7 Unscaled sources.

0.0.8 Theorem (n=3)

Let $a, d \equiv b - q_2 \in W^p_{s_0,\delta+2}$ $a, d \geq 0$, $s_0 \geq 0$, $\delta > -\frac{3}{p}$, $p > \frac{3}{2}$, be given on (M, γ), (p, σ, ρ) as. euc. positive Yamabe. The (L) equation

$$\triangle_\gamma \varphi - r(\gamma)\varphi + a\varphi^{-7} + d\varphi^5 = 0$$

has a solution $\varphi > 0$, with $1 - \varphi \in W^p_{s+2,\delta}$, $s = Inf(s_0, \sigma - 1)$, $-\frac{3}{p} < \delta < 1 - \frac{3}{p}$ if on M it holds that either

- 1. $d + a \leq r$, $\inf z_1(x) > 0$, and $\inf_{x \in M} z_2(x) \geq \max\{1, \sup_{x \in M} z_1(x)\}$, where $z_1(x)$ and $z_2(x)$ are the two positive roots of the polynomial

$$P_x(z) \equiv d(x)z^3 - r(x)z^2 + a(x).$$

- or 2.

$$||d||_{W^p_{0,\delta+2}} \leq \alpha, \qquad \alpha = C\{\frac{1}{1 + ||a||_{W^p_{0,\delta+2}}}\}^{1/4}$$

  $C > 0$ depends only on (M, γ).

Proof. 1. Super - subsolutions.

2. Iterate, starting from $u_0 = 1 - \varphi_0 = 0$. Then $u_n \geq 0$. Prove uniform bounds in $W^p_{2,\delta}$