On pages 141–225 of volume 88 of Acta Mathematica, which appeared in 1952, the following paper can be found

Y. Fourès-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires*

This is **the** funding paper for our current understanding of solutions of the Einstein equations.

Nous avons donc démontré le théorème suivant:

Étant donnée une solution $g_{\alpha\beta}$ du problème de Cauchy relativement aux équations $R_{\alpha\beta}=0$ (les données initiales satisfaisant sur S aux hypothèses précédemment énoncées de dérivabilité) il existe un changement de coordonnées, conservant S points par point, tel que les potentiels $\breve{g}_{\alpha\beta}$ dans le nouveau système de coordonnées vérifient partout les conditions d'isothermie et constituent la solution, unique, d'un problème de Cauchy, univoquement déterminé, relativement aux équations $G_{\alpha\beta}=0$.

Nous concluons donc, en termes de relativité:

Théorème. Il existe un espace-temps extérieur et un seul correspondant aux conditions initiales données sur S.

It seemed appropriate to Helmut and myself to mark the fiftieth birthday of this wonderful theorem by organising a summer school in which younger participants will learn from leading experts about recent developments in the field, while their older colleagues will have an opportunity to meet to exchange ideas and present their work. We thank all of you who helped us to make this happen by lecturing or organising the afternoon sessions.

During her whole career Yvonne kept contributing key results to a field which she helped to develop and grow. Her work has been a constant inspiration for the scientists entering the field.

We are pleased that mathematical relativity is growing, and that recently young people have been contributing deep new results, or solving long standing problems. We hope that this school will contribute to further the development of the field.

It is a pleasure to dedicate this school to Yvonne Choquet-Bruhat.

We feel very honored by her presence in Cargèse.

U(1) symmetry.

Spacetime $(V, {}^{(4)}g)$: principal fibre bundle with fiber $U(1) \equiv S^1$ and base $\Sigma \times R, \Sigma$ a smooth compact surface, oriented.

Spacetime 4 metric invariant under the S^1 action

$$^{(4)}g = e^{-2\gamma (3)}g + e^{2\gamma}(\theta)^2,$$

θ : 1-form on V represented in coordinates x^3 on the orbit of S^1, x^α on $\Sigma \times R$ by

$$\theta = dx^3 + A_\alpha dx^\alpha, \quad \alpha = 0, 1, 2$$

A , γ, ${}^{(3)}$g defined on $\Sigma \times R$.

$$^{(3)}g = -N^2 dt^2 + g_{ab}(dx^a + v^a dt)(dx^b + v^b dt)$$

N : lapse, v : shift.

$g_t \equiv g_{ab} dx^a dx^b$ riemannian metric on Σ_t.

Twist potential: ${}^{(4)}R_{3\alpha} = 0 \Rightarrow \exists\, \omega$

function ω such that

$$d\omega = e^{3\gamma} * dA$$

Einstein equations on V

Split as a coupled system:

A. Non linear hyperbolic system for γ, ω

$$g^{\alpha\beta}\nabla_\alpha\partial_\beta\gamma + \frac{1}{2}e^{-4\gamma}g^{\alpha\beta}\partial_\alpha\omega\partial_\beta\omega = 0 \qquad ({}^{(4)}R_{33}=0)$$

$$g^{\alpha\beta}\nabla_\alpha\partial_\beta\omega - 4g^{\alpha\beta}\partial_\alpha\gamma\partial_\beta\omega = 0 \qquad (d^2A=0)$$

wave map from $(\Sigma \times R, {}^{(3)}g)$ into Poincaré plane $(R^2, 2(d\gamma)^2 + (1/2)e^{4\gamma}(d\omega)^2)$.

B. Einstein equations on the 3-manifold $\Sigma \times R$ for the metric ${}^{(3)}g$, with source the wave map. $u \equiv (\gamma, \omega)$

$${}^{(3)}R_{\alpha\beta} = \rho_{\alpha\beta} \equiv \partial_\alpha u.\partial_\beta u$$

dot: scalar product in Poincaré plane.

equations equivalent to

- gauge conditions: equations for N and ν.
- constraints on each $\Sigma_t \equiv \Sigma \times \{t\}$ for g and extrinsic curvature k

$$k_{ab} \equiv -\frac{1}{2N}\bar{\partial}_0 g_{ab},$$

- Ordinary differential system to determine σ_{tf} (when constraints solved by conformal method)

Conformal method:

$$g_t \equiv e^{2\lambda}\sigma_t$$

Quasilinear elliptic system on each Σ_t, for λ and

$$h_{ab} = k_{ab} - \frac{1}{2} g_{ab}\tau$$

Mean extrinsic curvature

$$\tau \equiv g^{ab}k_{ab}$$

If Σ is compact

$$\frac{d}{dt}\int_{\Sigma_t} \mu_g = -\int_{\Sigma_t} N\tau\mu_g$$

The volume of Σ_t increases if $\tau < 0$. Take $\tau = \tau(t)$

$$D_a h^a_b = -\partial_b u . \dot{u} \quad , \quad \dot{u} \equiv e^{2\lambda} N^{-1}\partial_0 u$$

$$\Delta_\sigma \lambda = p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3$$

$$p_1 \equiv \frac{1}{4}\tau^2, p_2 \geq 0, p_3 \equiv \frac{1}{2}(R(\sigma) - |Du|^2)$$

$$p_2 = |\dot{u}|^2 + \frac{1}{2}|h|^2$$

$$R(\sigma) = -1 \Rightarrow p_3 < 0$$

Equations for lapse and shift

$$\Delta_\sigma N - qN = -e^{2\lambda}\partial_t \tau \qquad ({}^{(3)}R_{00} = -D_0 u \cdot u)$$

$$q \equiv e^{-2\lambda}(|e|^2 + |\dot{u}|^2) + \frac{1}{2}e^{2\lambda}\tau^2 > 0$$

$$\tau = -\frac{1}{t} \qquad \Rightarrow \qquad N \leq 2$$

$$(*) \quad (L_\sigma n)_{ab} \equiv D_a n_b + D_b n_a - \sigma_{ab} D_c n^c = f_{ab}$$

$$n_a \equiv e^{-2\lambda}\nu_a \quad , \quad f_{ab} = 2Ne^{-2\lambda}h_{ab} + \partial_t \sigma_{ab} - \sigma_{ab}\sigma^{cd}\partial_t \sigma_{cd}$$

Teichmüller parameters : genus$(\Sigma) > 1$

Teichmuller space $\sim M_{-1}/\mathcal{D}_0 \quad \sim R^{6G-6}$

gauge choice: $\sigma \in$ chosen cross section $Q \mapsto \psi(Q)$

$$\sigma_t = \psi(Q(t))$$

Differential system for $Q(t)$ deduced from integrability condition of $(*)$ and verification of the evolution part of 3 dimensional Einstein, ${}^{(3)}R_{ab} = D_a u \cdot D_b u$.

Local existence.

Cauchy data on Σ_{t_0}:

1. A C^∞ riemannian metric σ_0 and a C^∞ tensor q_0 with zero trace and divergence in the metric σ_0.

2. $u_0 = u(t_0,.)$ and $\dot{u}_0 = (N^{-1}e^{2\lambda}\partial_0 u)(t_0,.)$

Theorem.

The Cauchy problem with data $(u_0, \dot{u}_0) \in H_2 \times H_1$ for the Einstein equations with S^1 isometry group has a solution such that
$u \in C^0([t_0,T),H_2) \cap C^1([t_0,T),H_1)$,
$\lambda, N \in C^0([t_0,T), W_3^p) \cap C^1([t_0,T), W_2^p)$, $1<p<2$,
$N>0$ if $T-t_0$ is small enough.

The Sobolev norms of this theorem are respective to the metric σ_t, uniformly equivalent to σ_0.

Scheme for global existence

Standard method: a priori estimates of the norms appearing in local existence theorem. Here necessary to prove also σ_t uniformly equivalent to σ_0 . Requires an appropriate decay in t of H_1 norms of $u'(t,.)$ and $Du(t,.)$. This decay will be a consequence of the expansion of the metric $g(t,.)$ of Σ_t together with the hypothesis Genus$(\Sigma) > 1$. Its proof requires elliptic estimates for N, λ and ν. but also the introduction of corrected energies.

$R(\sigma) = -1$, Σ of genus greater than 1.

Polarized case: $\omega \equiv 0$

Parameter time t linked with τ by $t = -\tau^{-1}$, t increases from $t = t_0 > 0$ to infinity when Σ_t expands from $\tau_0 < 0$ to zero.

Equation satisfied by N (${}^{(3)}R_{00} = \rho_{00}$) implies

$$0 \leq N \leq 2$$

Energy estimates.

$$E(t) \equiv \int_{\Sigma_t} (|D\gamma|^2 + |\gamma'|^2 + \frac{1}{2}|h|^2)\mu_g$$

$$\frac{d}{dt}E(t) = \tau \int_{\Sigma_t} N((|\gamma'|^2 + \frac{1}{2}|h|^2)\mu_g \leq 0$$

$$E^{(1)}(t) \equiv \int_{\Sigma_t} (|\ \Delta_g\gamma\ |^2 + |\ D\gamma'\ |^2\ )\mu_g$$

$$\frac{dE^{(1)}}{dt} - 2\tau E^{(1)} = \tau \int_{\Sigma_t} N\ |\ D\gamma'\ |^2\ \mu_g + Z \leq Z$$

with Z bounded by $C_\sigma(E(t) + \tau^{-2}E^{(1)}(t))^{3/2}$, modulo elliptic estimates on $N-2$, $\partial_a N$, h, λ and their space derivatives in terms of the energies.

The number C_σ depends on the conformal metric σ. The bounds obtained on the energies are not sufficient to insure that σ remains uniformly equivalent to σ_0.

Elliptic estimates

$$\mathcal{E}^2 \equiv E(t) \quad , \quad \mathcal{E}_1^2 \equiv \tau^2 E^{(1)}(t)$$

1) $N \leq 2$ $\qquad e^{2\lambda} \geq 2\tau^{-2}$

2) Supposing $\mathcal{E}^2 + \mathcal{E}_1^2 \leq C$, some positive number

$$\frac{1}{\sqrt{2}} |\tau| e^{\lambda M} \leq 1 + C_\sigma (\mathcal{E}^2 + \mathcal{E}_1^2)$$

$$\| h \|_{L^\infty(g_t)} \leq C_\sigma |\tau| \{ \mathcal{E} + (\mathcal{E} + \mathcal{E}_1)^2 \}$$

$$0 \leq 2 - N \leq C_\sigma (\mathcal{E}^2 + \mathcal{E}\mathcal{E}_1)$$

$$\| \partial N \|_{L^\infty(g_t)} \leq C_\sigma |\tau| (\mathcal{E}^2 + \mathcal{E}\mathcal{E}_1)$$

give uniform bounds of energies <u>if</u> C_σ uniformly bounded.

Teichmuller parameters.

To prove that all space constants C_{σ_t} are uniformly bounded with $\sigma_t \equiv \psi(Q(t))$ we require that σ_t remains in some cross section of M_{-1} over T_{eich}.

We prove that $Q(t)$ remains in a compact subset of T_{eich} by using the energy of the harmonic map homotopic to identity $\Phi : (\Sigma, \sigma) \to (\Sigma, s), s$ some given metric on Σ of scalar curvature -1:

$$D(\sigma) \equiv \int_\Sigma \sigma^{ab} \frac{\partial \Phi^A}{\partial x^a} \frac{\partial \Phi^B}{\partial x^b} s_{AB}(\Phi) \mu_\sigma$$

$$= \int_\Sigma g^{ab} \partial_a \Phi^A \partial_b \Phi^B s_{ab}(\Phi) \mu_g$$

If $D(\sigma)$ remains in a bounded set of R the equivalence class of σ remains in a bounded set of T_{eich}. (Eells and Sampson).

A harmonic map is an extremal of the energy of the mappings between given riemannian manifolds, hence

$$\frac{d}{dt}D(\sigma) = \int_{\Sigma_t} \{\bar{\partial}_0 \; g^{ab}\partial_a\Phi^A\partial_b\Phi^B$$

$$- N\tau g^{ab}\partial_a\Phi^A\partial_b\Phi^B\} s_{AB}(\Phi)\mu_g$$

it implies

$$\frac{d}{dt}D(\sigma) \leq t^{-1}CC_\sigma[\varepsilon + (\varepsilon + \varepsilon_1)^2]D(\sigma)$$

If the energies decay it will be obtained under the a priori bounds hypothesis made ~~show that~~ there exists a number M_D depending only on the bounds in these hypothesis such that

Corrected energies.

For better decay. Exploit the negative (non definite) terms in the energies equalities.

Corrected first energy:

$$E_\alpha(t) = E(t) - \alpha\tau \int_{\Sigma_t} (u - \bar{u}).u' \mu_g$$

$$\bar{u} = \frac{1}{Vol_{g_t}} \int_{\Sigma_t} u\mu_g$$

Corrected second energy.

$$E_\alpha^{(1)}(t) = E^{(1)}(t) + \alpha\tau \int_{\Sigma_t} \Delta_g u.u' \mu_g$$

Poincaré inequality on (Σ, σ), function f

$$\|f - \bar{f}\|_{L^2(g)} \le e^{\lambda_M} \|f - \bar{f}\|_{L^2(\sigma)} \le e^{\lambda_M} \Lambda_\sigma^{-1/2} \|Df\|$$

with $\|\ .\ \|$ the L^2 norm on (Σ, σ), λ_M an upper bound of λ and Λ_σ the first positive eigenvalue of the operator $-\Delta \equiv -\Delta_\sigma$ acting on functions with mean value zero.

$$E(t) \le KE\alpha(t)\ ,\ \ K = \frac{1}{1 - a_t}$$

if

$$a_t \equiv \frac{\alpha|\tau|e^{\lambda_M}}{\Lambda_{\sigma_t}^{\frac{1}{2}}} < 1$$

Suppose

$$\Lambda_{\sigma_t} \geq \Lambda(1+\delta)^2,\ \Lambda > 0,\ \delta > 0$$

$$C(\varepsilon^2 + \varepsilon\varepsilon_1) < \frac{1}{2}\delta \quad , \quad \varepsilon^2 = E(t)$$

$$\varepsilon_1^2 = \tau^{-2} E^{(1)}(t)$$

$$\tau = -\frac{1}{t}$$

Take

$$\alpha \leq \frac{\Lambda^{\frac{1}{2}}}{\sqrt{2}} \text{ then } 1 - a_t \geq \frac{\delta}{2(1+\delta)}$$

Estimates

Estimates of h, N, λ lead to, when $R(\sigma) < 0$,

$$\frac{dE_\alpha}{dt} - k\tau E_\alpha \leq |\tau| A,$$

$$\frac{dE_\alpha^{(1)}}{dt} - (2+k)\tau E_\alpha^{(1)} + |\tau|^3 B$$

$$A, B \leq C\varepsilon(\varepsilon^2 + \varepsilon\varepsilon_1)$$

$$\Lambda \geq \frac{1}{8}, \alpha = \frac{1}{4}, \quad k = 1$$

$$\Lambda < \frac{1}{8}, \alpha < \frac{4}{8+\Lambda^{-1}}, \quad 0 < k < 1$$

Decay of total energy.

Total corrected energy.:

$$y(t) \equiv E_\alpha(t) + \tau^{-2} E_\alpha^{(1)}$$

Proven:

$$\frac{dy}{dt} = \frac{k}{t}[-y + My^{\frac{3}{2}}]$$

with M bounded by bounds of energies $(\varepsilon^2 + \varepsilon_1^2)$ and "constants" C_σ.

Implies

$$(\varepsilon^2 + \varepsilon_1^2)(t) \leq t^{-k} M(\varepsilon^2 + \varepsilon_1^2)(t_0)$$

Gives bounds by bootstrap argument.

Non linear stability theorem

Let $(\sigma_0, q_0) \in C^\infty(\Sigma_0)$ and $(u_0, \dot{u}_0)$ $\in H_2(\Sigma_0, \sigma_0) \times H_1(\Sigma_0, \sigma_0)$ be initial data for the polarized Einstein equations with U(1) isometry group on the initial manifold $\Sigma_0 \times U(1)$; suppose that σ_0 is such that $R(\sigma_0) = -1$.Then there exists a number $\eta > 0$ such that if

$$x_0 \equiv E_{tot}(t_0) < \eta$$

these Einstein equations have a solution on $\Sigma \times S^1 \times [t_0, \infty)$, with initial values determined by $\sigma_0, q_0, u_0, \dot{u}_0$. The spacetime is globally hyperbolic, future timelike and null geodesically complete.

Note: solution with zero energy,$R(\sigma) = -1$

$$-4dt^2 + 2t^2\sigma + (dx^3)^2$$

Our solution is asymptotic to such a solution as $t \equiv -\frac{1}{\tau}$ tends to + infinity.