Introduction to the Constraint Equations

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Topics: Submanifolds & the Constraints

Local existence → sufficiency

ADM & the Constraints

Linearisation Stability

Solving the Constraints - overview

Quasi-Spherical construction.

SOYCPEE

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Hypersurface $M^3 \overset{i}{\hookrightarrow} V^4$ Lorentz spacetime, metric g^V

M is <u>spacelike</u> $\iff g^V|_{TM} > 0$

$\iff$ induced metric $g = g^M = i^* g^V$ is Riemannian.

$\iff$ timelike (future, unit) normal n

X, Y tangent vector fields to M

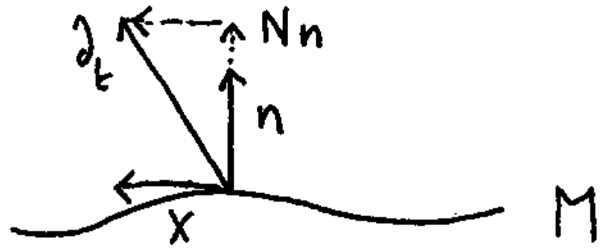
$$\left.\begin{aligned} \nabla^V_X Y &= \text{tangent} + \text{normal} \\ &= \nabla^M_X Y + K(X,Y)\, n \end{aligned}\right\} \quad \begin{aligned} &\nabla^M = \text{induced connection} \\ &K = 2^{\text{nd}} \text{ fundamental form} \end{aligned}$$

$$K(X,Y) = g^V(\nabla^V_X n, Y) = K(Y,X) \qquad = \text{extrinsic curvature.}$$

$$= -g^V(\nabla^V_X Y, n)$$

In adapted local coordinates (x,t) (so $M = \{t=0\}$)

$$n = N^{-1}(\partial_t - X^i\partial_i)$$



N = lapse

$X = X^i\partial_i$ = shift $\Rightarrow$ $\partial_t = Nn + X$

$$g^V = -N^2dt^2 + g^M_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$$

$$K_{ij} = K(\partial_i, \partial_j) = \tfrac{1}{2}N^{-1}\left(\partial_t g^M_{ij} - \nabla^M_i X_j - \nabla^M_j X_i\right)$$

GAUSS EQN: relate curvatures on M to K & V-curvature:

$$R^V(X,Y,Z,W) = g^V((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z, W) \qquad (\nabla = \nabla^V)$$

decompose $\nabla^V = \nabla^M + K$, for X, Y, Z, W tangent to M.

$$\Rightarrow \quad R^V(X,Y,Z,W) = R^M(X,Y,Z,W) + K(X,W)K(Y,Z) - K(X,Z)K(Y,W)$$

$$\Leftrightarrow \quad R^V_{ijkl} = R^M_{ijkl} + K_{il}K_{jk} - K_{ik}K_{jl} \quad \text{in index form}$$

take 2 traces via g^M-orthonormal frame $\{e_i\}_{i=1}^3$ gives

$$\sum_{i,j=1}^3 R^V(e_i, e_j, e_j, e_i) = R^M + (tr_M K)^2 - \|K\|^2$$

$$\Leftrightarrow \quad 2\,G(n,n) = R^M + (tr_M K)^2 - \|K\|^2$$

where $G_{\alpha\beta} = Ric^V_{\alpha\beta} - \frac{1}{2} R^V g^V_{\alpha\beta}$ is the Einstein tensor of g^V.

CODAZZI EQN:

$$R^{\nu}(X,Y,n,W) = g^{\nu}((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})n, W)$$

$$= \nabla_X K(Y,W) - \nabla_Y K(X,W)$$

$$\Leftrightarrow \quad R^{\nu}_{ijnk} = K_{jk;i} - K_{ik;j} \quad \text{in index form}$$

taking a trace gives

$$G(e_j, n) = \nabla_{e_i} K(e_j, e_i) - D_{e_j}(\mathrm{tr}_M K) \quad \Leftrightarrow \quad G_{jn} = K_{ij}{}^{;i} - K_{;j}$$

Assuming Einstein $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ gives the <u>Constraint equations</u>

$$\left.\begin{cases} R^M + (\mathrm{tr}\,K)^2 - \|K\|^2 = 16\pi T_{nn} \\ \nabla_j K_i{}^j - \nabla_i \,\mathrm{tr}\,K = 8\pi T_{ni} \end{cases}\right\} \Rightarrow$$

<u>necessary</u> conditions (for (M, g^M, K) to arise from Einstein eq$^{\underline{ns}}$.)

Constraints are Sufficient (Y. Choquet-Bruhat)

Reduced Einstein: $\rho(g) := Ric(g) + \frac{1}{2}\mathcal{L}_\tau g = 0$ $\quad (\mathcal{L}_\tau g = \nabla_\alpha \tau_\beta + \nabla_\beta \tau_\alpha)$

$\rho(g) = 0$ is quasilinear hyperbolic when

$\tau_\alpha = \tau_\alpha(\varphi) \approx$ tension of diffeomorphism $\varphi : (M, g) \longrightarrow (N, h)$ (fixed).

$\approx \nabla^N D\varphi \quad \rightarrow \quad -g^{\beta\gamma}\left(\nabla^N_\beta g_{\gamma\alpha} - \frac{1}{2}\nabla^N_\alpha g_{\beta\gamma}\right)$

Bianchi II & $\rho(g) = 0 \Rightarrow \quad \Box_g \tau + Ric_g(\tau) = 0$

Now ensure $\tau(0) = 0$ by choosing $h(0)$, $\partial_t h(0)$ <u>or</u> choosing $\partial_t g_{0\alpha}$

Then $\tau(0) = 0 \Rightarrow -2Ric_{\alpha\beta} = \partial_\alpha \tau_\beta + \partial_\beta \tau_\alpha$ at $t=0$

$\Rightarrow \quad g^{00}\nabla^N_t \tau_\alpha = -2G^0_\alpha$ + vanishing

$\Rightarrow \quad \partial_t \tau(0) = 0$ iff $G_{n\alpha} = 0$, $\Rightarrow$ sufficiency of constraints

Constructions for the Constraint equations.

- Conformal method (Lichnerowicz, Choquet Bruhat, York)
  - see talk by Jim Isenberg
- Gluing techniques
  - see talks by Dan Pollack; Justin Corvino
- Thin Sandwich (Wheeler, Belasco-Ohanián, Gyular Fodor & RB)
- Quasi-spherical (RB, Gilbert Weinstein & Brian Smith, Jason Sharples)

QS idea: assume polar coordinate r on $\mathbb{R}^3$ or $S^2 \times \mathbb{R}$ such that

level sets Σ_r have induced metric $r^2 ds_\circ^2$

$\Rightarrow$ metric $g = u^2 dr^2 + \mathring{g}_{AB}(r d\theta^A + \beta^A dr)(r d\theta^B + \beta^B dr)$

Second Variation Formula:

$$R = 2 D_n H + 2K - H^2 - |\mathrm{II}|^2 - 2u^{-1}\Delta u$$

where $n = u^{-1}(\partial_r - r^{-1}\beta^A \partial_A)$; $H =$ mean curvature of $\Sigma_r = -u^{-1}(2 - \mathrm{div}_\circ \beta)$

$K = r^{-2}\mathring{K} = r^{-2}$ (Gauss curvature of Σ_r)

$\Delta = r^{-2}\Delta_\circ =$ Laplacian of Σ_r

$$\Rightarrow \quad 2 r \partial_r u = \gamma u^2 \Delta_\circ u + 2 D_\beta u + (1 + \gamma B) u - \gamma (1 - \tfrac{1}{2} R r^2) u^3$$

where $\gamma = 1/(1 - \frac{1}{2}\mathrm{div}_\circ \beta)$, $B = B(\beta, \nabla\beta, \nabla \mathrm{div}_\circ \beta)$

Choose $R \leq 2r^{-2}$, $\beta = \beta^A$ such that $\mathrm{div}_\circ \beta < 2$ (β otherwise arbitrary)

$\Rightarrow$ parabolic eqⁿ for u, with global existence theorems $(0 < r < \infty)$

(RB, JDG'93)

Decay conditions on β (as $r \to 0, \infty$)

$\Rightarrow$ complete, C^∞, asymptotically flat metric, with prescribed R.

Example:

choose $R = 0$, any β such that $\beta = 0$ for $0 \leq r \leq 1$,

$\Rightarrow$ Time-symmetric initial data on $\mathbb{R}^3$ which is AF, $m_{ADM} > 0$

and such that $g|_{B(1)}$ = flat.

(Construction extends to B-H boundaries)

(Full QS constraint eqns treated by Sharples)

ADM formulation of Einstein evolution

phase space $\mathcal{F}$ of data (g, K) in some Sobolev space

constraint functional $\quad \Phi(g,K) = \begin{bmatrix} R(g) + (tr_g K)^2 - \|K\|^2 \\ 2\left(\nabla^j K_{ij} - \nabla_i \, tr_g K\right) \end{bmatrix}$

ADM Hamiltonian $\quad \mathcal{H}_{ADM}(g,K) = -\int_M \xi^\alpha \Phi_\alpha(g,K) \, dv_g$

$\Rightarrow$ eq$^{\underline{ns}}$ of motion : $\quad \frac{d}{dt}\begin{bmatrix} g \\ \pi \end{bmatrix} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} D\Phi^*(g,\pi)\cdot\xi$

where $\xi = \xi^\alpha =$ *lapse-shift (arbitrary)*

$\pi = K_{ij} - tr K \, g_{ij} \quad =$ *conjugate momentum*

ξ Killing $\Rightarrow \xi \in \ker D\Phi^*$ and conversely.

"Linearisation Stability" $\sim$ Constraint set $\mathcal{C}_0 = \{(g,K) : \Phi(g,K) = 0\}$

is a Hilbert manifold.