Isometry groups of Lorentz manifolds

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50 years of the Cauchy problem in General Relativity

Cargèse,

July 29 - August 10, 2002

1. **The question:**

— A subquestion: Homogeneous (non-proper) spacetimes

— A super-question: $Diff(M)$-action on the space of Lorentz metrics

— Some motivations

— Conformal case

2.**Examples:**

— Warped products

- Non-compact cases: Constant curvature spacetimes
- Compact cases:

— Flat cases: tori, SOL

— $SL(2, R) = AdS_3$,

— Warped Heisenberg groups,

— Non-homogeneous examples

3. **Results: compact case** (answer to the subquestion...)

4. **Results: non-compact case** (partial answers)

5. **Results:: Super-question** (the 2-dimensionnal case

The Question:

(M, g) a Lorentz manifold,

$G = \text{Isom}(M, g)$

Question When is the action of G on M **essential**?

$\Longleftrightarrow$

When the G-action can not preserve a Riemanniann metric on M?

$\Longleftrightarrow$

When is the action of G on M **non proper** ?

The G action is proper if: $\forall\ K \subset M$ compact, the set

$$G_K = \{g \in G, gK \cap K \neq \emptyset\}$$

is compact

The compact case:

When is the isometry group of a **compact** *Lorentz manifold is* **non-compact***?*

The sub-question:

Homogeneous space: $M = G/H$, G a Lie group, and H a closed subgroup of G.

– We suppose everywhere that G acts faithfully on M, i.e. we can not simplify G/H to a smaller G'/H'

– G acts on the left on M: $(g, xH) \in G \times M \rightarrow (gx)H \in M$

– This action preserves some "rigid geometric structure".

– The homogeneous space is of Riemannian type (resp. Lorentzian...) if the G-action preserves a Riemannian (resp. Lorentz...) metric on M.

– Stabilizer $(1.H) = Ad(H) \subset Ad(G) \subset GL(\mathcal{G})$, $\mathcal{G} =$ Lie algebra of G

Remark The G-action is of Riemannian type $\iff$ the action is proper $\iff$ $\bar{(}Ad(H))$ is compact

(in general $\iff$ H is compact)

Sub-question Classify $M = G/H$ of Lorentzian type (i.e. the G-action preserves some Lorentz metric on G/H), with H non compact.

The super-question: $Diff(M)$-action on the space of Lorentz metrics

$Diff^k(M)$ acts on $Lor^{k-1}(M)$ = space of C^{k-1} Lorentz metrics on M.

Endow them with the Banach or Frechet topology (for $k = \infty$)

— It is known that $Diff(M)$ acts properly on $Rie(M)$ (space of Riemannian metrics).

— The quotient $Riem(M)/Diff(M)$ is Hausdorff = modular space.

— A function on M is a Riemannian invariant.

QUESTION: When is the $Diff(M)$-action on $Lor(M)$ proper?

— If $g \in Lor(M)$, Stabilizer(g) = Isom(g)

— The $Diff(M)$-action proper $\Longrightarrow \forall\ g \in Lor(M)$, Isom($g$) is proper. (i.e. the super-question $\Longrightarrow$ the question).

– Gromov: the difficulty in the global studying of Lorentz manifolds lies in the fact that $Lor(M)/Diff(M)$ does not exist (as a Hausdorff space).

Some motivations:

1. For the sub-question:

• The homogeneous Riemannian problem (trivial for our talk here):

• We know very few about non-Riemannian homogeneous space.

– The interest of the Lorentz case: it is the easiest Non-Riemannian problem.

• The homogeneous compact Lorentz problem: Find G a Lie group, and H a closed Lie subgroup of G, such:

C1. The action of G on G/H preserves a Lorentz metric.

C2. $M = G/H$ is compact.

Fact

1. If H is **discrete**, then:

– C1 is equivalent to that the Lie algebra $\mathcal{G}$ has an $Ad(H)$- Lorentz scalar product.

– C2 means (by definition) that H is a co-compact lattice in G.

Explanation: Left translate to G a Lorentz scalar product on $\mathcal{G}$ which is $Ad(H)$-invariant. The Lorentz metric on G is: G-left-invariant, and H-right invariant. Therefore, it passes to a G-invariant Lorentz metric on G/H.

(exercise: Where have we used discreetness of H?)

For the question

Conformal groups of Riemannian manifolds:

(M, g) Riemannian manifold,

A priori, Conf(M, g) does not preserve a metric.

Lichnerowitch conjecture solved by **Lelong-Ferrand and Obatta**: This happens only for the usual spheres and Euclidean spaces.

Remark There are analogous conjectures in geometric dynamics...

Examples: general constructions

• Products

N Lorentz, with Isom(N) essential, $\Longrightarrow M = N \times L$ has an essential isometry group.

• Local products:

$\tilde{N}$ Lorentz $M = \tilde{N} \times \tilde{L}/\Gamma$, where Γ is a **non split** subgroup of $Isom(\tilde{L}) \times Isom(\tilde{L})$

If the centralizer of Γ acts non-properly on $\tilde{N} \times \tilde{L}$, then M is essential.

• Same examples for the homogeneous Lorentz problem.

• The warped product construction (generalization of direct products):

(L, h) Riemannian manifolds

(N, g) Lorentz

$w : L \to R^+$ a (warping) function.

The warped product $M = L \times_w N$, is the topological product $L \times N$, endowed with the metric $h \oplus wg$.

• If $f : N \to N$, is an isometry then, the trivial extension: $\bar{f} : (x, y) \in L \times N \to (x, f(y)) \in L \times N$, is an isometry of $L \times_w N$

(In particular, in the class of Lorentz manifolds with large isometry groups, one can perform warped products by (any) Riemannian manifolds.)

• **Examples: compact spaces: 1. Flat case:**

Flat tori $= (R^n, g)/Z^n$,

g a Lorentz scalar product on R^n

Isom $(T^n, g) = T^n \rtimes O(g, Z)$,

$O(g, Z) = O(g) \cap GL(n, Z)$,

$O(g) =$ the orthogonal group of g ($\sim O(1, n-1)$)

• Dimension 2 (Avez):

$A \in SL(2, Z)$ hyperbolic,

ω^u (resp. ω^s) linear forms on R^2 defining, the stable and unstable foliations of A.

$g = \omega^u \omega^s$.

A preserves g

Isom$(T^2, g) =$ (essentially) $T^2 \rtimes Z$, Z generated by A.

• Dimension > 2

Harisch-Chandra Borel: if g is rational, then $O(g, Z)$ is a lattice in $O(g)$. (in particular $O(g, Z)$ is isomorphic to the fundamental group of a finite volume hyperbolic manifold)

• Suspension T^3_A

The suspension of A gives a flat manifold with an isometric flow which is Anosov (chaotic)

$T^3_A = SOL/\Gamma$, SOL the 3-dimensional unimodular solvable non-nilpotent group.

• **Examples: compact spaces: 2. Local** AdS_3 **space**

$G = SL(2, \mathbf{R})$

$\triangleright$ The Killing form k on the Lie algebra $sl(2, R)$ has signature $- + +$ (may be $- - +$, in this case, consider $-k$)

– It is Ad-invariant, in particular $Ad(H)$-invariant for any co-compact lattice.

Thus *For any co-compact lattice* $H \subset SL(2, \mathbf{R})$*,* $SL(2, \mathbf{R}/H$ *is a homogenous compact spacetime*

• Alternative explanation of the Fact above:

Right translate k , and get a right invariant Lorentz metric on G. Thus, it passes to right quotients G/H.

-G acts by the left G/H

– This action is isometric, since, also the left action on G on itself preserves the Lorentz metric, because k is bi-invariant.

Remark: $SL(2, \mathbf{R})/H$ is (up to 2-cover) to unit tangent bundle of the hyperbolic surface $\mathbf{H}^2/H$.

• Examples: compact case: 3. Warped Heisenberg groups

A family of sympathetic groups: Warped Heisenberg groups: family of **solvable** groups looking like $SL(2,R)$:

• they admit Lorentz bi-invariant Lorentz metrics, i.e. their Lie algebra admit Ad-invariant Lorentz scalar products ($\neq$ the Killing form, which is degenerate).

• they have co-compact lattices!

$\triangleright$ G is a warped Heisenberg group, H a lattice

$\triangleright$ As in the case of $SL(2,\mathbf{R})$,

$M = G/H$ is a compact homogeneous spacetime, where H is a cocompact lattice

(co-compact is superfluous, since any lattice in a solvable Lie group is co-compact).

• **The simplest example** of warped Heisenberg groups: dimension 4 (known as oscillator group, Diamond group...)

The semi-direct product $G = S^1 \ltimes Heis$

$Heis$ = Heisenberg group (of dimension 3):

$$Heis = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbf{R} \}$$

$Heis$ is characterized essentially, by the existence of a non-split exact sequence:

$1 \to \mathbf{R} \to Heis \to \mathbf{R}^2 \to 1$

———

$G = S^1 \ltimes Heis$ is defined by:

– S^1 acts trivially on the center $\mathbf{R}$, and acts by rotation on $\mathbf{R}^2$

—

• Also G is characterized by being a non-trivial central extension of Ec by S^1,

Ec = group of Euclidean isometries of the plane.

$1 \to S^1 \to G \to Ec \to 1$

Generalization: canonical warped Heisenberg groups

• Recall the construction of Heisenberg algebras: $\mathcal{HE}_d$ (dim $= 2d + 1$)

$\mathbf{R} \oplus \mathbf{C}^d$, with basis $Z, e_1, \ldots, e_d$

The only non-vanishing brackets are: $[e_k, ie_k] = Z$. (here $i = \sqrt{-1}$)

Equivalently,

$[X, Y] = \omega(X, Y)Z$.

ω symplectic form, i.e. $\omega(X, Y) =< X, iY >_0$,

$<, >_0$ hermitian product

• Canonical warped Heisenberg algebras

Add an exterior element t, such that:

$[t, e_k] = ie_k, [t, ie_k] = -e_k$, and $[t, Z] = 0$

• Scalar product

$<, >$

Endow $\mathbf{C}^d$ with its hermitian $<, >_0$

Decree $\mathbf{C}^d$ is orthogonal to $\{t, Z\}$.

$< t, t >=< Z, Z >= 0$ and $< t, Z >= 1$.

• $<, >$ is a Lorentz scalar product which is $Ad(\mathcal{HE}^t_d)$-invariant! i.e. $\forall u \in \mathcal{HE}^t_d$ u, ad_u is antisymmetric with respect to $<, >$

exercise: Why this doesn't work for the Heisenberg algebras themselves?

• Consider $\tilde{G} = \tilde{He}^t_d$ the simply connected Lie group generated by $\mathcal{HE}^t_d$

• $\tilde{He}^t_d$ is a semi-direct product of $\mathbf{R}$ by He_d:

- The action of $\mathbf{R}$ on the center is trivial.

- The action on $\mathbf{C}^d$ is via multiplication by $\exp is$.

- This is in fact an action of S^1

• Consider $G = He^t_d = \tilde{He}^t_d/\mathbf{Z} = S^1 \ltimes He_d$

• Any lattice in the Heisenberg group He_d is a lattice in He^t_d.

- example of a lattice in He_1:

$$Heis_{\mathbf{Z}} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbf{Z} \}$$

General construction of warped Heisenberg groups:

- Consider a semi-direct product $\mathbf{R} \ltimes He_d$, where $\mathbf{R}$ acts on $\mathbf{C}^d$ via:

$s \to \exp(2\pi s A) \in U(d)$ such that:

- $\exp(2\pi A) = 1$,
- A diagonalizable, $\lambda_1, \ldots, \lambda_d \in \mathbf{Z}$
- The λ_i have the same sign.

Thus, the action of $\mathbf{R}$ factors via an action of S^1

– The semi-direct product $G = S^1 \ltimes He_d$ is called a warped Heisenberg group.

– The conditions guaranty that G has a bi-invariant Lorentz metric.

– G has lattices

– Any quotient G/H, H a lattice, is a compact homogeneous spacetime.

Some remarks:

1. recall that the Lorentz scalar product was defined by: $\mathbf{C}^d$ is endowed with its hermitian metric$<, >_0$ and is orthogonal to the plane $\{t, Z\}$, $< t, t >=< Z, Z >= 0$ and $< t, Z >= 1$.

– One may take $< t, t >=$ Constant $\neq 0$, and multiply the other products by any constant ($\neq 0$), and gets another bi-invariant Lorentz metric.

– However, up to automorphism, there exists only one bi-invariant Lorentz metric on a warped Heisenberg group..

-In particular a metric is isometric to any multiple of itself. This follows from existence of homotheties. (This is true for $\mathbf{R}^n$ but not for $SL(2, \mathbf{R})$.

— Warped Heisenberg groups are (locally) symmetric Lorentz spaces of non reductive type.

– They have non-reductive holonomy, i.e. they have a codimension 1 **parallel** foliation which has no supplementary parallel direction field.

– The Ricci curvature of a warped Heisenberg group equals its Killing form (up to constant).

Historical comments

• He_1^t the 4-dimensional Heisenberg group is known as:

– Diamond group, in Representation Theory

– Oscillator group in Representation Theory and quantum mechanics

• The bi-invariant Lorentz metrics were known to medina-Revoy, and "partially" to Zimmer and Gromov.

– This seems folkloric in relativistic literature: some gravitational plane waves spacetimes...

– Witten-Nappi: "a WZW model based on a non semi-simple group" (1993)

Justification of the name "oscillator group":

The Lie algebra $\mathcal{HE}_1^t$ has the following representation in the algebra of operators of the Hilbert space $E = L^2(\mathbf{R})$:

$Z \to 1 =$ (Identity)

$X \to q$ (position)

$Y \to p$ (impulsion)

$t \to p^2 + q^2$ (energy),

where the operators q and p are given by:

$q(f) = xf \ (f \in L^2(\mathbf{R}))$

$p(f) = \frac{\partial f}{\partial x}$

— To show that this gives a homomorphism, one verifies in particular: $[q, p] = 1$, which is the Heisenberg uncertainty principle.

$p^2 + q^2$ is the energy of the harmonic oscillator, which explains the origin of the terminology.

Espaces symetriques ~~vo~~ reductible non decomposables :-

- Dim 2, 3 : Non
- Dim 4

$$g = (\alpha x^2 + \beta y^2)\, du^2 + 2\, du\, dv + dx^2 + dy^2$$

- (α, β) parametres : $\alpha^2 + \beta^2 = 1$
- $\frac{\partial}{\partial v}$: Champ de Killing //
- Ricci $= 0 \iff \alpha = -\beta = \frac{1}{2}$

Ondes gravitationnelles

- $$g = 2\, du\, dv + H(u, x, y)\, du^2 + dx^2 + dy^2$$

- Ricc $= 0 \iff \Delta_{xy} H = \frac{\partial^2}{\partial x^2} H + \frac{\partial^2 H}{\partial y^2} = 0$
- Ondes planes.

$$H(u, x, y) = f(u)(x^2 - y^2) + g(u)\, xy$$

Cas $\alpha = \beta = 1/2$

- Ric $\neq 0$, Weyl $= 0$: conf plat
- Groupe oscillateur = groupe diamond = groupe de Heisenberg tordu : (Medina-Revoy, Streater, Wallen, Nappi, Adams-Stich, Gromov, Zimmer, Z.....)
- $G = S^1 \ltimes$ Heis
  $1 \to \mathbb{R} \to G \to$ Euc $\to 1$
- Euc = groupe d'Euclide = $S^1 \ltimes \mathbb{R}^2$
- G = extension centrale de Euc
- $\exists$ métrique lorentzienne bi-invariante sur G

Sub-question: **Classification of compact homogeneous spacetimes: Part I: structure of their isometry group**

Theorem 1 *Let $M = G/H$ be a compact homogeneous Lorentz manifold.*

Then, up to compact objects: G is $SL(2, \mathbf{R})$ or a warped Heisenberg group.

More precisely: there is a subgroup $S \subset G$, such that:

- *S is normal, and the Lie algebra of S is a factor in $\mathcal{G}$*
- *S is co-compact in G (i.e. G/S is compact)*
- *S is isomorphic to $PSL_k(2, \mathbf{R})$ the k-folded cover of $PSL(2, \mathbf{R})$, or*

  -S is a warped Heisenberg group.
- *S acts on M locally freely (i.e. stabilizer in S are discrete)*

Corollary The stabilizer H is **"almost discrete":** its connected component is compact. *(This is non-obvious a priori, and false for non-compact homogeneous spacetimes, and for general homogeneous pseudo-Riemannian manifolds, even compact)*

Subquestion: **Classification of compact homogeneous spacetimes: Part II: their geometric structure**

Theorem 2 *Up to compact objects, they are isometric to* S/H*, where* H *is a co-compact lattice (in particular discrete) in* S*, which is* $PSL(2, \mathbf{R})$ *or a warped Heisenberg group.*

- Roughly, M *is a a "local product" modeled on* $S \times \tilde{L}$*, where* $\tilde{L}$ *is a homogeneous Riemannian manifold*

—– Details:

- *The case* $S = PSL_k(2, \mathbf{R})$ *(due to Gromov)*

- $M = S \times \tilde{L}/H$*:*

- $\tilde{L}$ *is a compact homogeneous Riemannian manifold*

- There is H_0 *a lattice in* S*, such that* H *is the graph of a homomorphism* $\rho : H_0 \to Isom(\tilde{L})$

- The centralizer of $\rho(H_0)$ *acts transitively on* $\tilde{L}$*.*

- The metric on $S \times \tilde{L}$ *is:* $c.Killing \otimes r_{\tilde{L}}$*,* c *constant,* $r_{\tilde{L}}$ *the Riemannian metric of* $\tilde{L}$

- Conversely, with these data, one constructs a compact homogeneous spacetime.

- S *a warped Heisenberg group: a little bit complicated description...*

The question (non-homogeneous case)

Theorem 3 *(Zimmer, Gromov, Adams-Stuck, Zeghib) Recall that the Lie algebra of a compact Riemannian manifold is a sum of an abelian Lie algebra with a semi-simple Lie algebra of compact type. (i.e. the Lie algebra of a compact semi-simple Lie group).*

– In the Lorentz case, the new factor that might occur, is a subalgebra of $\mathcal{S}$, where $\mathcal{S}$ is the Lie algebra of $SL(2, \mathbf{R})$ or a warped Heisenberg group.

More details

(Killing algebra of M = the Lie algebra of its isometry group)

Theorem 4 *(Adams-Stuck, Z.) The Killing Lie algebra of a compact Lorentz manifold is isomorphic to a direct sum*

$$\mathcal{K} + \mathbf{R}^k + \mathcal{S},$$

where $\mathcal{K}$ is the Lie algebra of a compact semi-simple Lie group, $k \geq 0$ is an integer and $\mathcal{S}$ is trivial or isomorphic to:

- ▷ *a Heisenberg algebra (of some dimension),*
- ▷ *a warped Heisenberg algebra, or*
- ▷ $sl(2, \mathbf{R})$*.*

Conversely, any such algebra is isomorphic to the Lie algebra of the isometry group of some compact Lorentz manifold.

The non-compact case

— Trivial counter-example: G a Lie group, endow $\mathcal{G}$ with any Lorentz scalar product, and left translate it on G. The left G action is isometric $\rightarrow G$ is a homogeneous spacetime.

But, the G-action is proper (stabilizer are trivial).

The same is true, in general, for Isom(G) (which might cantain properly G)

The subquestion: Find H closed, **non-compact**, such that $Ad(H)$ preserves a Lorentz scaler product on $\mathcal{G}/\mathcal{H}$?

Example: Symmetric spaces (reductive or not)

— Non-reductive Lorentz symmetric spaces: clssified by Cahen-Parker.

Non-compact spacetimes are more interesting in physics.

Observation: Only few classical exact solutions have essential isometry groups.

One may try to prove:

"*a physical solution (i.e. a natural energy-implusion tensor + causality conditions* $\Longrightarrow$ *the solution has a non-essential isometry group, unless, it is very special (e.g. -Minkowski, dS, AdS...)* "

Results: Non-comapct case

"First work": Nadine Kowalsky

Thesis with Zimmer,

Notice in CRAS with 8 Theorems,

Article in Ann. Maths: Proofs of 4 Theorems,

Unfortunately, she prematurely dead...

Algebraic hypothesis:

G a simple (sometimes semi-simple) connected Lie group, acting isometrically non-properly on a Lorentz manifold M.

Principal algebraic result: $G = O(1, n)$ or $O(2, n)$

Geometric result (without proof) M is essentially dS_n or AdS_n.

More exactly, this is true up to (a local) warped product.

Works by S. Adams (also an old student of Zimmer):

– New proof of Kowalsky's algebraic Theorem.

– New groups, but with a stronger dynamical hypothesis.

Super-question The $Diff(M)$-action on $Lor(M)$

– Pierre Mounoud (Lafontaine's student)

– Case of compact surfaces: Klein Bottle or a torus

Theorem 1 For M = Klein bottle, the Diff(M)-action on $Lor(M)$ is proper.

Torus case: M

$\mathcal{F}$ the space of flat metrics on M.

– Such a metric is linear (on $\mathbf{R}^2$)

– The $Diff_0(M)$-action on $\mathcal{F}$ is proper

– The $Diff(M)$-action on $\mathcal{F}/Diff_0(M)$ is identified to the action of $SL(2,\mathbf{Z})$ on

$$SL(2,\mathbf{R})/\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbf{R}\}$$

which is "dual" to the action of the geodesic flow on the modular surface $\mathbf{H}^2/SL(2,\mathbf{Z})$.

– This is in particular ergodic

Theorem 2 The $Diff(M)$-action on $Lor(M) - \mathcal{F}$ (the space of non-flat metrics) is proper.

An amazing lemma: If a Lorentz metric on the torus has curvature constant along one isotropic foliation, then this metric is flat.