On the expanding direction of Gowdy vacuum spacetimes

- Motivation, general context.
- Definition of Gowdy spacetimes.
- Results.
- Idea of proofs.

Expanding direction, division of results

Small data results:

- Open set of initial data.

- Close to known solutions.

Cases with symmetry:

- "Empty" set of initial data.

- Arbitrary initial data within a symmetry class.

Definition

- Vacuum solution of Einstein's equations.

- Compact spatial Cauchy surfaces.

- The metric is invariant under an effective action of $U(1) \times U(1)$ on the Cauchy surfaces.

- The twist constants vanish.

One of the twist constants:

$$\kappa_X = \epsilon_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta.$$

The symmetry assumption leads to the conclusion that the Cauchy surfaces have topology T^3, $S^2 \times S^1$, S^3 or one of the Lens spaces.

Metric and equations

Metric:

$$g = t^{-1/2} e^{\lambda/2}(-dt^2 + d\theta^2) \qquad (1)$$

$$+t[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + (e^P Q^2 + e^{-P}) d\delta^2],$$

where $t \in \mathbb{R}_+$ and $(\theta, \sigma, \delta) \in T^3$. Einstein's equations:

$$P_{tt} + \frac{1}{t} P_t - P_{\theta\theta} - e^{2P}(Q_t^2 - Q_\theta^2) = 0 \qquad (2)$$

$$Q_{tt} + \frac{1}{t} Q_t - Q_{\theta\theta} + 2(P_t Q_t - P_\theta Q_\theta) = 0, \qquad (3)$$

and

$$\lambda_t = t[P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)] \qquad (4)$$

$$\lambda_\theta = 2t(P_\theta P_t + e^{2P} Q_\theta Q_t). \qquad (5)$$

Wave map structure

Let

$$g_0 = -dt^2 + d\theta^2 + t^2 d\phi^2 \quad \text{on} \quad \mathbb{R}_+ \times T^2$$

and

$$g_1 = dP^2 + e^{2P} dQ^2 \quad \text{on} \quad \mathbb{R}^2.$$

Then (2)-(3) are the wave map equations for a map from $(\mathbb{R}_+ \times T^2, g_0)$ to $(\mathbb{R}^2, g_1)$ which is independent of the ϕ variable on T^2.

Note that $(\mathbb{R}^2, g_1)$ is isometric to the upper half plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric

$$g_H = \frac{dx^2 + dy^2}{y^2}$$

under $(Q, P) \mapsto (Q, e^{-P}) = (x, y) = \mathbf{x}$.

Results

Numerically, it was observed that

$$l(t) = \int_{S^1} \sqrt{P_\theta^2 + e^{2P} Q_\theta^2} d\theta$$

decays like $t^{-1/2}$. Another quantity of interest is

$$H = \frac{1}{2} \int_{S^1} [P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)] d\theta.$$

One can prove the following

Theorem 1 *Consider a solution to (2)-(3). Then there is a* $T > 0$ *and a* c_H *such that*

$$|tH(t) - c_H| \leq K \frac{\ln t}{t}$$

for all $t \geq T$. *Furthermore* $c_H = 0$ *if and only if the solution is independent of* θ.

Conserved quantities

Due to the symmetries of hyperbolic space, the following quantities are conserved:

$$
\begin{aligned}
\alpha &= \frac{1}{2\pi}\int_{S^1}\{2Q(tQ_t)e^{2P} - 2(tP_t)\}d\theta \\
\beta &= \frac{1}{2\pi}\int_{S^1} e^{2P}(tQ_t)d\theta \\
\gamma &= \int_{S^1}\frac{1}{2\pi}\{(tQ_t)(1 - e^{2P}Q^2) + 2Q(tP_t)\}d\theta.
\end{aligned}
$$

The expression

$$\alpha^2 + 4\beta\gamma$$

is invariant under the isometries.

Spatially homogeneous solutions

For a spatially homogeneous solution

$$\alpha^2 + 4\beta\gamma = 4t^2(P_t^2 + e^{2P}Q_t^2) \geq 0.$$

Orbit:

A point if $\alpha^2 + 4\beta\gamma = 0$.

A geodesic if $\alpha^2 + 4\beta\gamma > 0$.

General solutions

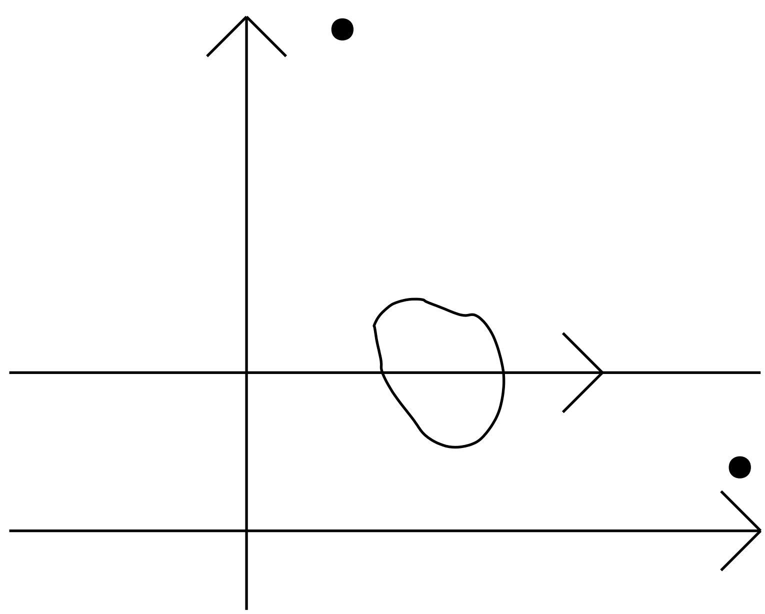
Theorem 2 *Consider a solution to (2)-(3). If* $\alpha = \beta = \gamma = 0$, *there is an* $\mathbf{x}_0 \in H$ *and a* $T > 0$ *such that if* $\mathbf{x} = (Q, e^{-P})$,

$$d_H[\mathbf{x}(t, \theta), \mathbf{x}_0] \leq Kt^{-1/2}$$

for all $t \geq T$.

Theorem 3 *Consider a solution to (2)-(3) with* $\alpha^2 + 4\beta\gamma = 0$, *but for which not all the constants are zero. After applying an isometry to the solution*

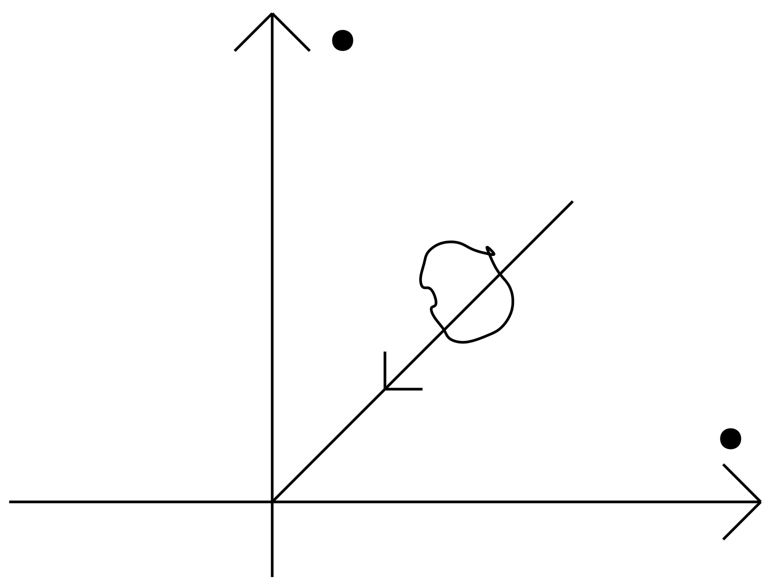
$$\begin{aligned} \|y - c_1\|_{C(S^1,\mathbb{R})} &\leq Kt^{-1/2} \\ \|x - \ln t - c_2\|_{C(S^1,\mathbb{R})} &\leq Kt^{-1/2}. \end{aligned}$$



Theorem 4 *Consider a solution to (2)-(3) with $\alpha^2 + 4\beta\gamma > 0$. After applying an isometry to the solution*

$$
\begin{aligned}
\|\frac{x}{y} - c_1\|_{C(S^1,\mathbb{R})} &\leq Kt^{-1/2} \\
\| \ln y + \delta \ln t + c_2\|_{C(S^1,\mathbb{R})} &\leq Kt^{-1/2},
\end{aligned}
$$

where $\delta = \sqrt{\alpha^2 + 4\beta\gamma}/2$.

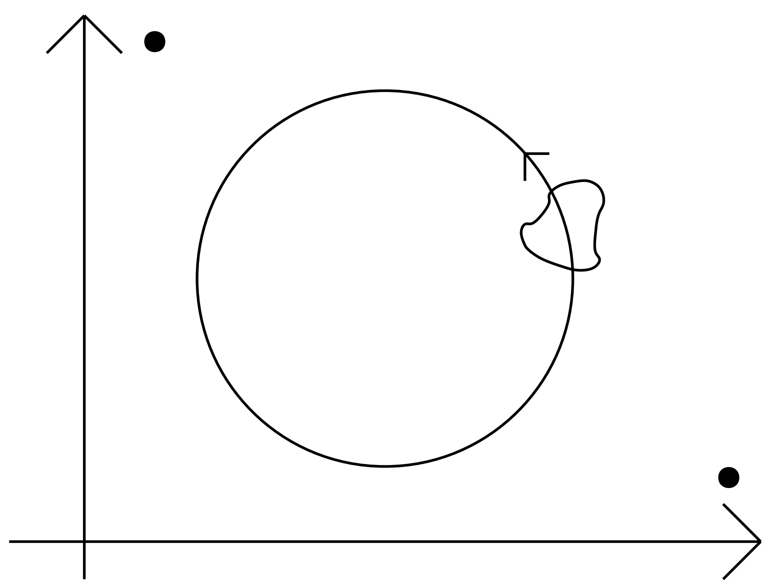


Theorem 5 *Consider a solution to (2)-(3) such that* $\alpha^2 + 4\beta\gamma < 0$. *There is a* K, $T > 0$ *and a circle* Γ *such that for every* $t_0 \geq T$ *there is a curve* γ_{t_0} *with the properties*

$$\gamma_{t_0}(t) \in \Gamma, \quad d_H[\mathbf{x}(t,\theta), \gamma_{t_0}(t)] \leq K t_0^{-1/2}$$

$$\gamma_{t_0}[t_1 \exp(\frac{2\pi}{\delta})] = \gamma_{t_0}(t_1), \quad |\gamma_{t_0}'(t)| = \frac{r\delta}{t},$$

for $t \geq t_0$, *where* $2\pi r = l(\Gamma)$, $\delta = \sqrt{|\alpha^2 + 4\beta\gamma|}/2$.



The remaining function

Theorem 6 *Consider a solution to (2)-(3). Then, for solutions that are not independent of* θ*,*

$$\|\lambda(t) - c_\lambda t\|_{C(S^1,\mathbb{R})} \leq K(\ln t)^2$$

where $c_\lambda > 0$*.*

Proof. By (4)-(5), we have $|\lambda_\theta| \leq \lambda_t$, thus

$$\|\lambda - <\lambda>\|_{C(S^1,\mathbb{R})} \leq \int_{S^1} |\lambda_\theta| d\theta \leq$$

$$\leq \int_{S^1} \lambda_t d\theta = 2tH(t) \leq K.$$

Since $tH(t)$ converges, we also have

$$|<\lambda_t> - \frac{c_H}{\pi}| \leq K \frac{\ln t}{t},$$

where c_H is positive under the assumptions of the theorem. We get the conclusion of the theorem. □

Control of the sup norm

Proposition 1 *Consider a solution to (2)-(3). Then*

$$\|P_t\|_{C(S^1,\mathbb{R})} + \|P_\theta\|_{C(S^1,\mathbb{R})} + \|e^P Q_t\|_{C(S^1,\mathbb{R})} +$$

$$+ \|e^P Q_\theta\|_{C(S^1,\mathbb{R})} \leq K t^{-1/2}.$$

Theorem 7 *Consider a metric given by (1), where λ, P and Q are solutions to (2)-(5). Let $\gamma : (s_-, s_+) \to \mathbb{R}_+ \times T^3$ be an inextendible causal geodesic with respect to this metric and assume that $< \gamma', \partial_t >< 0$. Then γ is future complete.*

Energy decay, example 1

Consider the ODE

$$\ddot{x} + 2a\dot{x} + b^2 x = 0,$$

where $a > 0$ and $b^2 > a^2$. Let

$$H = \frac{1}{2}(\dot{x}^2 + b^2 x^2).$$

Compute

$$\frac{dH}{dt} = -2a\dot{x}^2.$$

Introduce a correction term

$$\Gamma = ax\dot{x}.$$

Then

$$|\Gamma| = |\frac{a}{b}||bx\dot{x}| \leq |\frac{a}{b}|\frac{1}{2}(\dot{x}^2 + b^2 x^2) = |\frac{a}{b}|H.$$

so that $c_1 H \leq H + \Gamma \leq c_2 H$. Furthermore

$$\frac{d(H+\Gamma)}{dt} = -2a(H + \Gamma).$$

Energy decay, example 2

Consider

$$P_{tt} + \frac{1}{t} P_t - P_{\theta\theta} = 0.$$

The natural energy is

$$H = \frac{1}{2} \int_{S^1} (P_t^2 + P_\theta^2) d\theta.$$

We have

$$\frac{dH}{dt} = -\frac{1}{t} \int_{S^1} P_t^2 d\theta.$$

Correction

$$\Gamma = \frac{1}{2t} \int_{S^1} (P - < P >) P_t d\theta$$

Note that $|\Gamma| \leq KH/t$. Furthermore,

$$\frac{d\Gamma}{dt} = -\frac{2}{t}\Gamma - \frac{\pi}{t} < P_t >^2 + \frac{1}{2t} \int_{S^1} (P_t^2 - P_\theta^2) d\theta,$$

$$\frac{d(H + \Gamma)}{dt} \leq -\frac{1}{t}(H + \Gamma) + \frac{K}{t^2} H.$$

Energy decay, small data

$$H = \frac{1}{2}\int_{S^1}[P_t^2 + P_\theta^2 + e^{2P}(Q_t^2 + Q_\theta^2)]d\theta.$$

Compute

$$\frac{dH}{dt} = -\frac{1}{t}\int_{S^1} P_t^2 d\theta - \frac{1}{t}\int_{S^1} e^{2P} Q_t^2 d\theta.$$

Define

$$\Gamma_1 = \frac{1}{2t}\int_{S^1}(P - < P >)P_t d\theta.$$

We have

$$\frac{d\Gamma_1}{dt} \leq -\frac{2}{t}\Gamma_1 + \frac{1}{2t}\int_{S^1}(P_t^2 - P_\theta^2)d\theta +$$

$$+\frac{1}{2t}\int_{S^1}(P - < P >)e^{2P}(Q_t^2 - Q_\theta^2)d\theta$$

Also

$$\Gamma_2 = \frac{1}{2t}\int_{S^1} e^{2<P>}(Q - < Q >)Q_t d\theta.$$

Energy decay, small data

If

$$\Gamma = \Gamma_1 + \Gamma_2,$$

then

$$|\Gamma| \leq \frac{K}{t} H$$

and

$$\frac{d(H + \Gamma)}{dt} \leq -\frac{1}{t}(H + \Gamma) - \frac{1}{t}\Gamma + \frac{K}{t} H^{3/2}.$$

Thus there is an $\eta > 0$ such that if there is a $t_0 > 0$ such that $H(t_0) \leq \eta$, then

$$H(t) \leq \frac{K}{t}$$

for all $t \geq t_0$.

Energy decay, large data

Note that

$$\frac{1}{t}\int_{S^1} P_t^2 d\theta + \frac{1}{t}\int_{S^1} e^{2P} Q_t^2 d\theta \in L^1([t_0,\infty)).$$

One can show that

$$\int_{t_0}^{t} \frac{1}{s}\int_{S^1} e^{2P}(Q_t^2 - Q_\theta^2) d\theta ds$$

converges. Thus

$$\frac{1}{t}\int_{S^1} e^{2P} Q_\theta^2 d\theta \in L^1([t_0,\infty)).$$

Using this information, one can prove that

$$\int_{t_0}^{t} \frac{1}{s}\int_{S^1} (P_t^2 - P_\theta^2) d\theta ds$$

converges. Thus

$$\frac{1}{t} H \in L^1([t_0,\infty)),$$

so that $H \to 0$.

Behaviour of the mean values

Preservation of α yields

$$t < P_t >= \beta < Q > -\frac{\alpha}{2}$$

$$+\frac{1}{2\pi}\int_{S^1} te^{2P}(Q- < Q >)Q_t d\theta,$$

Preservation of β yields

$$te^{<P>} < Q_t >= \beta e^{-<P>} + O(1)$$

and preservation of γ yields

$$te^{<P>} < Q_t >= e^{<P>}[\gamma + \alpha < Q > -\beta < Q >^2]$$

$$+O(1).$$

Behaviour of the mean values

Lemma 1 *Consider a solution to (2)-(3) with* $\beta \neq 0$*. If* $\alpha^2 + 4\beta\gamma \geq 0$ *then* $< Q >$ *is bounded and* $< P >$ *is bounded from below for* $t \geq T$*. If* $\alpha^2 + 4\beta\gamma < 0$*,* $< P >$ *and* $< Q >$ *are both bounded for* $t \geq T$*.*

Thus

$$< P_t >^2 + e^{2<P>} < Q_t >^2 \leq \frac{K}{t^2}.$$

A model equation

Lemma 2 *Consider a solution to (2)-(3). Then*

$$\int_{t_0}^{t}[< P_t > -\frac{\beta}{s} < Q > +\frac{\alpha}{2s}]ds = O(t_0^{-1/2}).$$

Proof. Consider

$$\int_{t_0}^{t}\int_{S^1} e^{2P}(Q- < Q >)Q_t d\theta ds.$$

Observe that

$$\int_{S^1} e^{2P}(Q- < Q >) < Q_t > d\theta = O(t^{-3/2})$$

Consequently

$$\int_{S^1} e^{2P}(Q- < Q >)Q_t d\theta$$

$$= \int_{S^1} e^{2P}(Q- < Q >)(Q_t- < Q_t >)d\theta + O(t^{-3/2}).$$

Thus

$$\int_{t_0}^{t}\int_{S^1} e^{2P}(Q- < Q >)Q_t d\theta ds$$

$$= \frac{1}{2}\int_{t_0}^{t}\int_{S^1} e^{2P}\partial_t(Q- < Q >)^2 d\theta ds +$$

$$+O(t_0^{-1/2}) = [\frac{1}{2}\int_{S^1} e^{2P}(Q- < Q >)^2 d\theta]_{t_0}^{t} -$$

$$-\int_{t_0}^{t}\int_{S^1} P_t e^{2P}(Q- < Q >)^2 d\theta ds$$

$$+O(t_0^{-1/2}) = O(t_0^{-1/2}).$$