Definition: Convergence of Metrics. Let $L^{k,p} = L^{k,p}(\Omega) =$ Sobolev space: k weak derivatives in L^p, $\Omega \subset \mathbb{R}^n$.

A sequence of metrics g_i on manifolds M_i converges in $L^{k,p}$ topology to limit metric g on manifold M if $\exists$ charts $\Phi = \{\phi_k\}$, covering M, and diffeomorphisms $F_i : M \to M_i$, s.t.

$$(F_i^* g_i)_{\alpha\beta} \to g_{\alpha\beta}, \tag{1.1}$$

in the $L^{k,p}$ topology.

Same definition holds for convergence in the $C^{k,\alpha}$ topology, as well as the weak $L^{k,p}$ topology.

(Recall $f_i \to f$ weakly in $L^p(\Omega)$ iff $\int f_i g \to \int fg$, $\forall g \in L^q(\Omega)$, $p^{-1} + q^{-1} = 1$.)

To obtain local control on a metric, or sequence of metrics, assume

curvature bounds.

Cheeger-Gromov theory requires a bound on the full Riemann curvature tensor

$$|Riem| \leq K, \tag{1.2}$$

for some $K < \infty$. Since {# components of Riem} >> {# components of metric}, (in dimensions ≥ 4), this is overdetermined set of constraints on the metric and so overly restrictive.

Much more natural to impose bounds on the Ricci curvature

$$|Ric| \leq k. \tag{1.3}$$

Of course, bounds on Ricci natural in general relativity, via the Einstein equations. So emphasize (1.3) over (1.2) whenever possible.

To obtain local control, first important step: choose

"good gauge" = local coord. system for g

Look at coordinates built from the geometry of the metric itself.

(i) geodesic normal coordinates

(ii) distance coordinates

Entail loss of derivatives: 2 or 1 resp.

Optimal choice:

(iii) harmonic coordinates

Definition: harmonic radius r_h. Fix a function topology, say $L^{k,p}$ and a constant $C > 1$. Given $x \in (M, g)$, the $L^{k,p}$ harmonic radius

$$r_h(x) = r_h^{k,p}(x)$$

= largest radius s.t. on ball $B_x(r_h(x))$ have harmonic coordinate chart $U = \{u_\alpha\}$ in which the metric $g = g_{\alpha\beta}$ is controlled in $L^{k,p}$ norm: thus,

$$C^{-1}\delta_{\alpha\beta} \leq g_{\alpha\beta} \leq C\delta_{\alpha\beta}, \quad \text{(as bilinear forms)}, \tag{2.1}$$

$$[r_h(x)]^{kp-n} \int_{B_x(r_h(x))} |\partial^k g_{\alpha\beta}|^p dV \leq C. \tag{2.2}$$

$kp > n$, so that $L^{k,p} \subset C^0$. Estimates are scale invariant, (rescale harmonic coordinates also), so r_h scales as a distance.

Lower bound on injectivity radius can be considerably weakened. Define the 1-cross $Cro_1(x)$ of (M, g, x) by

$$Cro_1(x) = sup\{t : \gamma_x(t) = \text{min. geod. on } [-t, t], \gamma(0) = x\}.$$

Then set

$$Cro_1(M, g) = \inf_x Cro_1(x).$$

Has natural analogue in Lorentzian geometry —replace minimizing geodesic by maximizing time-like geodesic.

Theorem 2.2 Convergence II. *Let M be a 4-manifold. Then space of metrics s.t.*

$$|Ric| \leq k, \; Cro_1 \geq c_o, \; vol \geq v_o, \; diam \leq D. \tag{2.10}$$

is precompact in $C^{1,\alpha}$ and weak $L^{2,p}$.

The proof is the same as Theorem 2.1. Lower bound on Cro_1 implies that on blow-up limit $(N, \bar{g}_\infty)$ above, have a line. The splitting theorem then again implies that the limit is flat $\mathbb{R}^4$, giving the same contradiction.

Need volume bound to obtain limit $\mathbb{R}^4$ instead of some quotient $\mathbb{R}^4/\Gamma$ flat manifold.

Of course, in dimension 3 any Ricci-flat manifold is flat. So have $C^{1,\alpha}$ precompactness within the class of metrics on 3-manifolds satisfying

$$|Ric| \leq k, \; vol \geq v_o, \; diam \leq D. \tag{2.11}$$

Volume comparison theorem (Bishop-Gromov):

If $Ric \geq (n-1)k$, for some $k > -\infty$, then

$$\frac{volB_x(r)}{volB_k(r)} \downarrow \tag{2.12}$$

$volB_k(r) =$ volume of r-ball in space form of const. curv. k.

In particular, obtain lower bound on volumes of balls on all scales:

$$volB_x(r) \geq \frac{volM}{volB_k(D)} \cdot volB_k(r), \tag{2.13}$$

$D = diamM$.

Cheeger: if bound on Ric strengthened to

$$K_P \geq -K,\ vol \geq v_o,\ diam \leq D, \tag{2.14}$$

where $K_P =$ sectional curvature of any plane P, then

$$inj_g(M) \geq i_o(K, v_o, D).$$

However, this estimate fails under bounds on Ricci.

Example 2.4 Eguchi-Hanson metrics

Let g_λ = family of Eguchi-Hanson metrics on TS^2:

$$g_\lambda = [1 - (\frac{\lambda}{r})^4]^{-1} dr^2 + r^2[1 - (\frac{\lambda}{r})^4]\theta_1^2 + r^2(\theta_2^2 + \theta_3^2), \qquad (2.15)$$

for $r \geq \lambda > 0$.

Locus $r = \lambda$ = image of the 0-section: totally geodesic round $S^2(\lambda)$ of radius λ.

The metrics g_λ are Ricci-flat, and are all homothetic, i.e. are rescalings (via diffeomorphisms) of a fixed metric; in fact,

$$g_\lambda = \lambda^2 \cdot \psi_\lambda^*(g_1), \qquad (2.16)$$

where $\psi_\lambda(r) = \lambda r$. As

$$\lambda \to 0,$$

i.e. as one blows down the metrics,

$$(TS^2, g_\lambda) \to (C(\mathbb{RP}^3), g_0);$$

g_0 = singular flat metric.

Convergence is smooth where $r \geq r_o$, any fixed $r_o > 0$, but is not smooth at $r = 0$. For any $x \in S^2(\lambda)$,

$$inj_{g_\lambda}(x) \to 0.$$

However, volumes of unit balls remains uniformly bounded below.

Metrics g_λ converge to space with orbifold singularity $\mathbb{R}^4/\mathbb{Z}_{\neq}$.

No $C^{1,\alpha}$, or even C^0, compactness.

Large class of Ricci-flat ALE (asymptotically locally Euclidean) spaces, whose metrics are asymptotic to cones $C(S^3/\Gamma)$, $\Gamma \subset SO(4)$, on spherical space forms. This is the family of

ALE gravitational instantons

Gibbons-Hawking, Hawking's Euclidean quantum gravity program.

Theorem 2.5 Convergence III. *Let $\{g_i\}$ be a sequence of metrics on a 4-manifold M, satisfying*

$$|Ric| \leq k, \ vol \geq v_o, \ diam \leq D. \tag{2.17}$$

Then, on a subsequence,

$$(M, g_i) \to (V, g). \tag{2.18}$$

(V, g) = *orbifold, with finite number of singular points* $\{q_j\}$.

• Each q has neighborhood homeomorphic to cone $C(S^3/\Gamma)$, for $\Gamma \subset SO(4)$.
• Metric g is $L^{2,p}$ on regular set

$$V_0 = V \setminus \cup\{q_j\}.$$

•. g extends in a local uniformization of q to C^0 metric on B^4.
• Embeddings $F_i : V_0 \to M$ s.t.

$$F_i^*(g_i) \to g,$$

in weak $L^{2,p}$.
• Convergence in (2.18) is in Gromov-Hausdorff topology, i.e. convergence as metric spaces.

Some important issues in proof:

• Chern-Gauss-Bonnet formula implies

$$\frac{1}{8\pi^2}\int_M |R|^2 dV \leq \chi(M) + C(k, V_o),$$

$C(k, V_o)$ depends only on Ricci curvature bound k and an upper bound V_o on $vol_g M$.

• For each singular point $q \in V$, $\exists$ sequence of rescalings

$$\bar{g}_i = \lambda_i^2 g_i, \quad \lambda_i \to \infty,$$

and base points $x_i \in M$, $x_i \to q$, s.t. on a subsequence

$$(M, \bar{g}_i, x_i) \to (N, \bar{g}_\infty, x_\infty),$$

in $L^{2,p}$, where

$$(N, \bar{g}_\infty) = \text{non-trivial Ricci-flat ALE space.}$$

Any such ALE space has a definite amount of curvature in L^2,

$$\int_N |R|^2 \geq c_0.$$

So only a finite number of such singular points.

Further, the ALE spaces N are embedded in M, in a topologically essential way.

Near singularities, (M, g_i) resembles blow-downs of ALE spaces.

§3. Collapse/Formation of Cusps.

Issue: Behavior of metrics g_i when curvature bounded and

$$vol_{g_i} B_{x_i}(1) \to 0.$$

Throughout this section, assume

$$dim M = 3.$$

Examples. (Product collapse)
$M = S^1 \times V$. Consider curve of metrics

$$g_\lambda = \lambda^2 (d\phi)^2 + g_V.$$

As $\lambda \to 0$, collapse to V.

$Riem_{g_\lambda} = Riem_{g_V}$, so

$$|Riem_{g_\lambda}| \leq K, \text{ as } \lambda \to 0. \tag{3.1}$$

blowing down metric in one direction

(Berger collapse)
On $M = S^3$, consider curve of metrics

$$g_\lambda = \lambda^2 \theta_1^2 + (\theta_2^2 + \theta_3^2).$$

Isometric S^1 action; $L(\text{orbit}) = 2\pi\lambda$.

$$\lambda \to 0 \Rightarrow S^1 \text{ orbit } \to \{pt\}.$$

Main Point:

$$|Riem_{g_\lambda}| \leq K, \text{ as } \lambda \to 0.$$

Collapse to limit space S^2 with bdd. curvature.

Behavior of Taub-NUT metric at Cauchy horizon.

Exactly same procedure works on any n-manifold with

free or locally free S^1.

A Seifert fibered space N = 3-manifold with locally free S^1 action; fibers over surface V with S^1 fibers.

Definition 3.1 A graph manifold G is a closed 3-manifold obtained by glueing Seifert fibered spaces by toral automorphisms of the boundary tori.

Thus,

$$G = S \cup L,$$

S = union of Seifert fibered spaces
L = union of $T^2 \times I$ - glueing region.

On any graph manifold, can construct metrics invariant under S^1/T^2 actions.

Consequence Any compact graph manifold G has volume collapse with bounded curvature:

$$|Ric_{g_i}| \leq k, \quad vol_{g_i}G \to 0. \tag{3.2}$$

If $G = S$, Seifert fibered, can collapse with bounded diameter, $diam_{g_i}S \leq D$. However, if decomposition non-trivial, then

$$diam_{g_i}N \to \infty, \tag{3.3}$$

$N =$ any S or L component. This behavior special to dim 3.

Cheeger-Gromov theory implies converse holds:

Theorem 3.2 *If M is a closed 3-manifold which collapses with bounded curvature, i.e. (3.2) holds, then M is a graph manifold.*

Idea of Proof. First,

$$vol_{g_i}B_x(1) \to 0 \Rightarrow inj_{g_i}(x) \to 0.$$

At any x, rescale inj to size 1,

$$\bar{g}_i = [inj_{g_i}(x)]^{-2} \cdot g_i.$$

Now $|Riem_{\bar{g}_i}| \sim 0$. Thus, metrics $\bar{g}_i$ close to flat metrics on $\mathbb{R}^3/\Gamma \sim \mathbb{R}^2 \times S^1, \mathbb{R} \times S^1 \times S^1$.

Local geometry = geometry on scale of inj, modeled by *non-trivial, flat* 3-manifolds. Glue these local structures consistently.

• **Unwrapping collapse.**

N = Seifert fibered space, (possibly with boundary). Then

$$\pi_1(S^1) \hookrightarrow \pi_1(N),$$

unless $N = S^3/\Gamma$ or $N = D^2 \times S^1$.

Hence, G a graph manifold, not S^3/Γ and no solid torus in $S \cup L$ decomposition, then

$$\pi_1(orbits) \hookrightarrow \pi_1(N).$$

This means can pass to covering spaces, unwrapping orbits, to unwrap collapse. Obtain convergence to limits in covers.

Further, limits have free isometric S^1 or T^2 action.

Extra symmetry.

Again, this unwrapping special to dim 3.

Formation of cusps.

Mixture of convergence/collapse. Given complete Riemannian manifold (M, g), choose $\varepsilon > 0$ small. Set

$$M^\varepsilon = \{x \in M : volB_x(1) \geq \varepsilon\}, M_\varepsilon = \{x \in M : volB_x(1) \leq \varepsilon\}.$$

$$M^\varepsilon = \varepsilon\text{-thick part}, \ M_\varepsilon = \varepsilon\text{-thin part}.$$

Now let $\{g_i\}$ = sequence of complete metrics on M.

- If $x_i \in M^\varepsilon$, then have convergence in regions about x_i.
- If $y_i \in M_{\varepsilon_o}$, then region about y_i is graph manifold.
- If $z_i \in M_{\varepsilon_i}$, $\varepsilon_i \to 0$, then region about z_i collapsing.

Theorem 3.3 Cusp Formation *M a closed 3-manifold and g_i sequence of metrics on M satisfying*

$$M^\varepsilon \neq \emptyset, \quad M_\varepsilon \neq \emptyset, \quad \forall \varepsilon.$$

Then pointed subsequences (M, g_i, p_i) converge to:

- *complete cusps N - open 3-manifolds with graph manifold ends.*
- *collapsed graph manifolds of infinite diameter.*