Rotating black holes and disks as explicitly solvable boundary value problems

Gernot Neugebauer
Theoretisch-Physikalisches Institut
der Universität Jena
Max-Wien-Platz 1
D-07743 Jena
neugebauer@tpi.uni-jena.de

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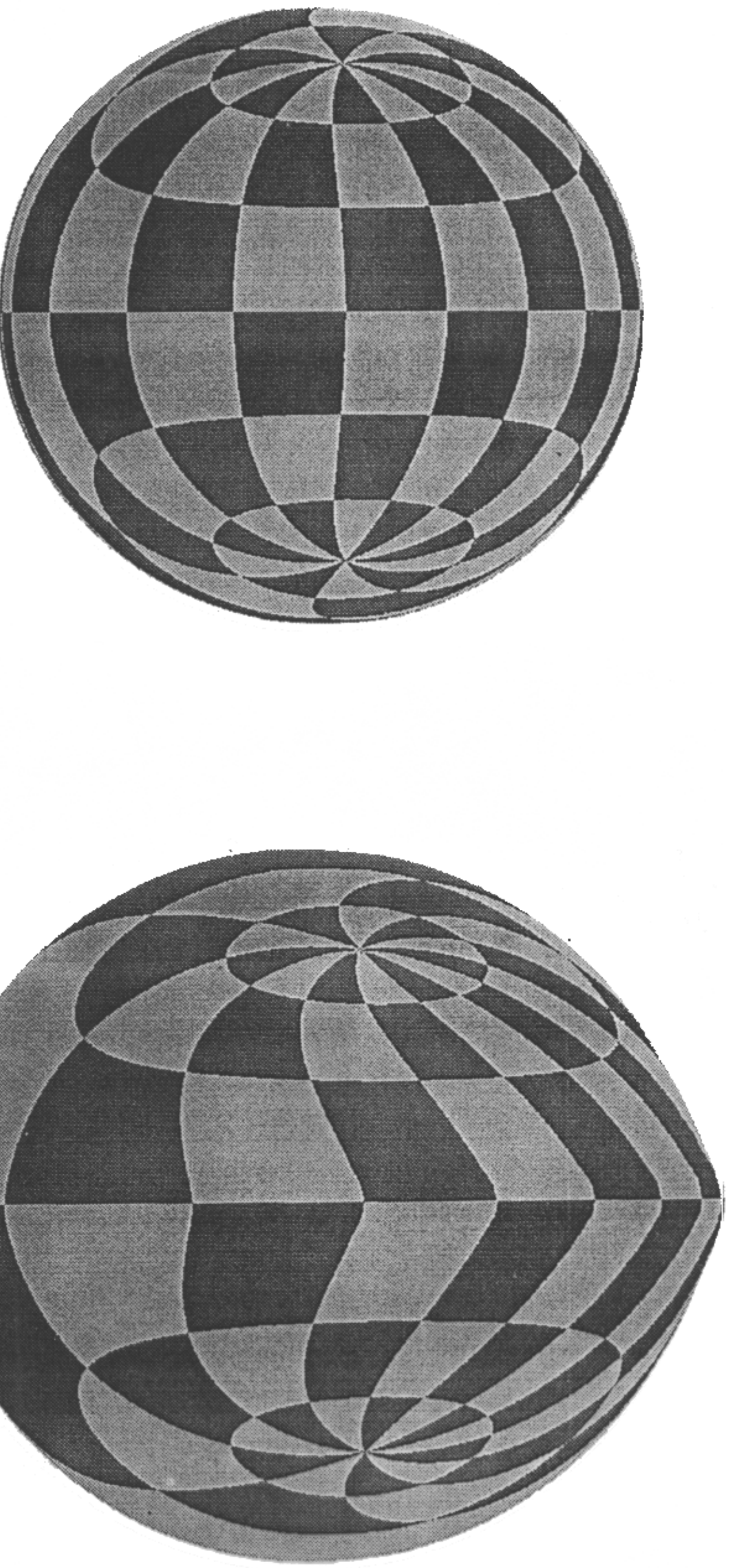
1. Introduction



4D ray-tracing pictures of neutron stars
(cf. Herold, H. and Neugebauer, G., Gravitational fields of rapidly rotating neutron stars, Lecture Notes in Physics 410, (Springer 1992), pp.305-340)

Physical problem

Uniformly rotating bodies (stars)

$\rightarrow$

Axisymmetry: azimuthal Killing vector η, $\eta^2 > 0$

Stationarity: timelike Killing vector ξ, $\xi^2 < 0$

$\curvearrowright$ G_2 (2-dimensional group of motions)

Mathematical task

Solve free boundary value problems to the axisymm. stationary Einstein eqs. in vacuo

FEqs.: $R_{ik} = 0$; $\eta^i = \delta^i_\varphi$ $\xi^i = \delta^i_t$:

$$ds^2 = e^{-2U}\left[e^{2k}(d\varrho^2 + d\zeta^2) + \varrho^2 d\phi^2\right] - e^{2U}(dt + a\,d\phi)^2$$

$$U = U(\varrho, \zeta) \;,\; a = a(\varrho, \zeta) \;,\; k = k(\varrho, \zeta)$$

Boundary Values:

① ∞: Minkowski space; $a = 0, k = 0, U = 0$

② Axis of symmetry: $a = 0,\ k = 0$

③ Surface of the body: data depend on the problem

Reformulation : $f := e^{2U} + ib \; , \; b = b\{a\}$

"Ernst eqs."

FEqs: $\mathrm{Re} f \, \Delta f = (\nabla f)^2 \; , \; f = f(\rho, \zeta)$

BV : ① ∞ : $f = 1$

② Axis : regularity of f

③ Surface of the body : data depend on the body's structure

Algorithm :

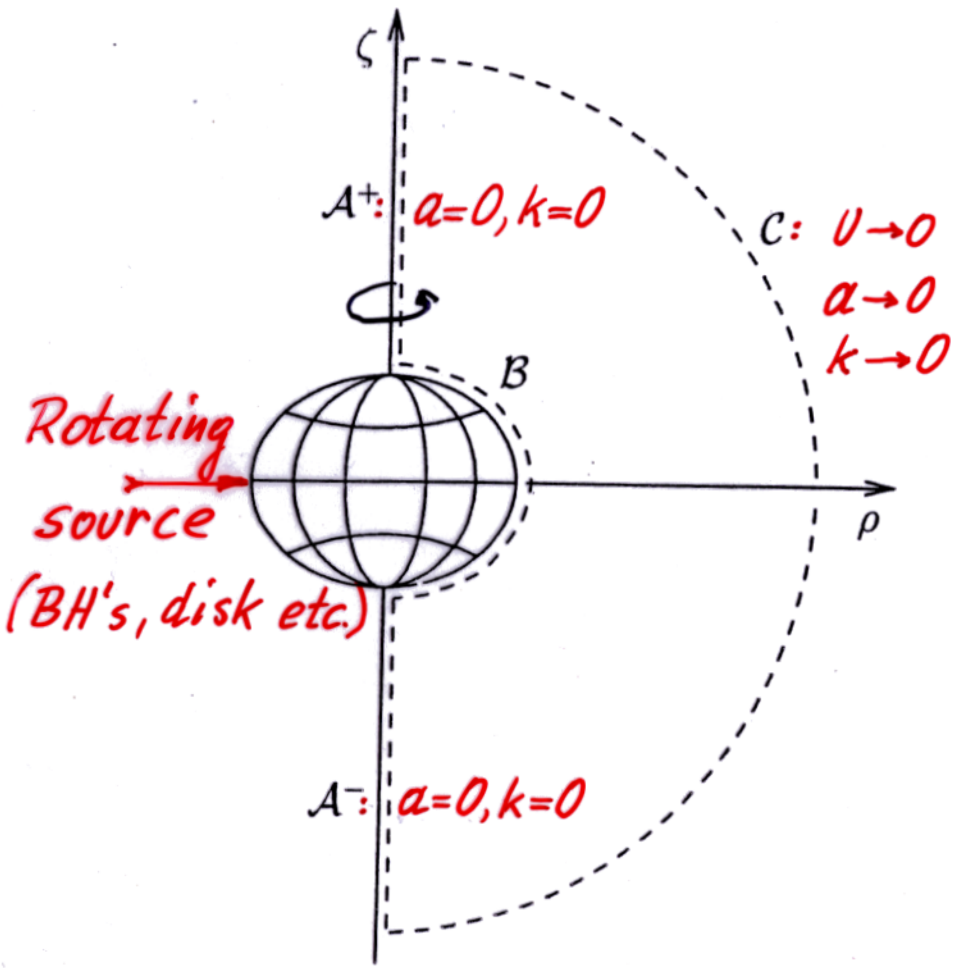
$$\begin{tikzcd} \underline{f(\rho,\zeta)} \arrow[rr, leftrightarrow, "\text{line int.}"] \arrow[dr, "\text{line int.}"] & & (U, a) \arrow[dl, "\text{line int.}"'] \\ & k & \end{tikzcd}$$

Note: The Ernst equation holds likewise in the co-rotating framework (primed!)
$t' = t \; , \; \phi' = \phi - \Omega t \quad (\Omega = \text{constant})$:

$$\underline{\mathrm{Re} f' \Delta f' = (\nabla f')^2}$$

2. Boundary value problem and Inverse Method (IM)

Boundary value problem



Field equations and Linear problem (LP)

$$\boxed{\Re(f)\Delta f = (\nabla f)^2 \quad , \qquad f = e^{2U} + ib.}$$

$$\updownarrow$$

$$\Phi_{,z} = \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} + \lambda \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \right\} \Phi,$$

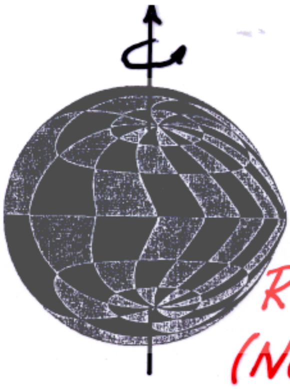
$$\Phi_{,\bar z} = \left\{ \begin{pmatrix} \bar A & 0 \\ 0 & \bar B \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar A \\ \bar B & 0 \end{pmatrix} \right\} \Phi$$

$$A = \frac{f_{,z}}{f + \bar f}, \; B = \frac{\bar f_{,z}}{f + \bar f}, \; \lambda = \sqrt{\frac{K - i\bar z}{K + iz}}$$

$$z = \rho + i\zeta$$

Examples

Analytic construction by IM:



Rotating (Neutron) star

possible?

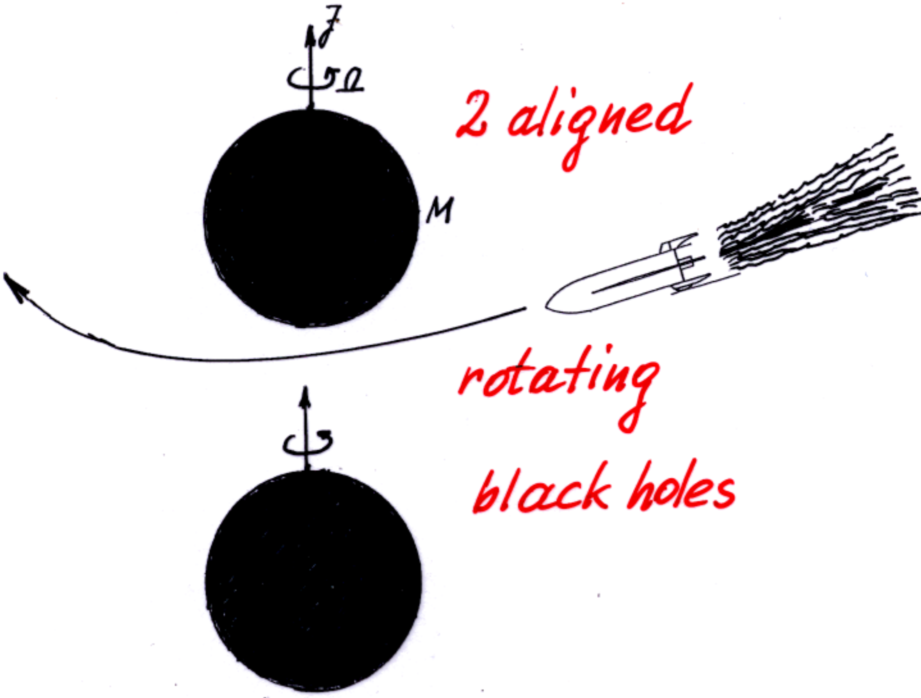
no!

(surface values of the star do not completely reflect its internal structure)

Rotating black hole

yes! → this talk

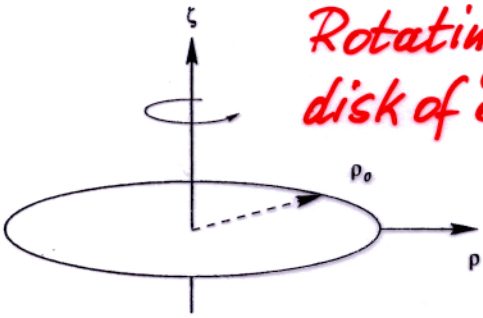
(horizon determines the solution completely)



2 aligned rotating black holes

yes! → this talk

(2 separated horizons, spin-spin repulsion compensates gravitational attraction: no-go proof)



Rotating disk of dust

yes! → this talk

(global solution of a rotating body problem, 'galaxy' model, "testbed")

Black hole in the center of a (ring) disk

should be possible

(AGN model: Galactic black hole)

Normalization of $\phi(\lambda, \rho, \zeta) \leftrightarrow \phi(K, \rho, \zeta)$

① $$\phi = \begin{pmatrix} \psi(\lambda,\rho,\zeta) & \psi(-\lambda,\rho,\zeta) \\ \chi(\lambda,\rho,\zeta) & -\chi(-\lambda,\rho,\zeta) \end{pmatrix}$$

"lives" in the 2 sheets of the Riemann surface ($\lambda^2 = \frac{K - i\bar{z}}{K + iz}$)

② $$\chi(\lambda,\rho,\zeta) = \overline{\psi\left(\frac{1}{\bar{\lambda}}, \rho, \zeta\right)}$$

③ $$\phi(\lambda = 1, \rho, \zeta) = \begin{pmatrix} \overline{f(\rho,\zeta)} & 1 \\ f(\rho,\zeta) & -1 \end{pmatrix}$$

$$\curvearrowright \quad f(\rho,\zeta) = \phi_{21}(\lambda = 1, \rho, \zeta)$$

$$\downarrow$$

$$ds^2$$

Idea of the Inverse Method

Note: We apply the Inverse Method to elliptic PDE's!

Programme: Integrate the Linear Problem along $\mathcal{A}^+ C \mathcal{A}^- \mathcal{B}$ (dashed line) picking up the available information ($\mathcal{B}$: boundary values, $\mathcal{A}^\pm$: regularity, C: $f=1$, Minkowski space)

Result: ① $\mathcal{A}^\pm$: $\phi(K,\zeta)$ as a holomorphic function in K ($\rightarrow$ zeros, poles, jumps etc.)

② Holomorphic structure allows extension of $\phi(K,\zeta)$ to $\phi(K,\varrho,\zeta)$ (via $\lambda = \sqrt{\frac{K-i\bar{z}}{K+iz}}$)

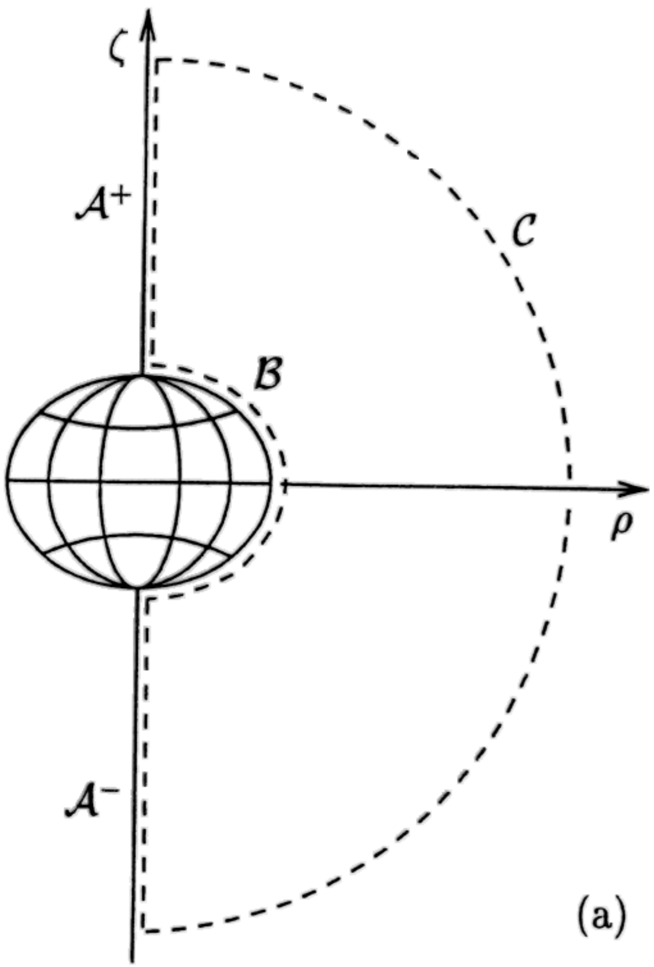
③ $\phi(K,\varrho,\zeta) \longrightarrow f(\varrho,\zeta) \longrightarrow ds^2$

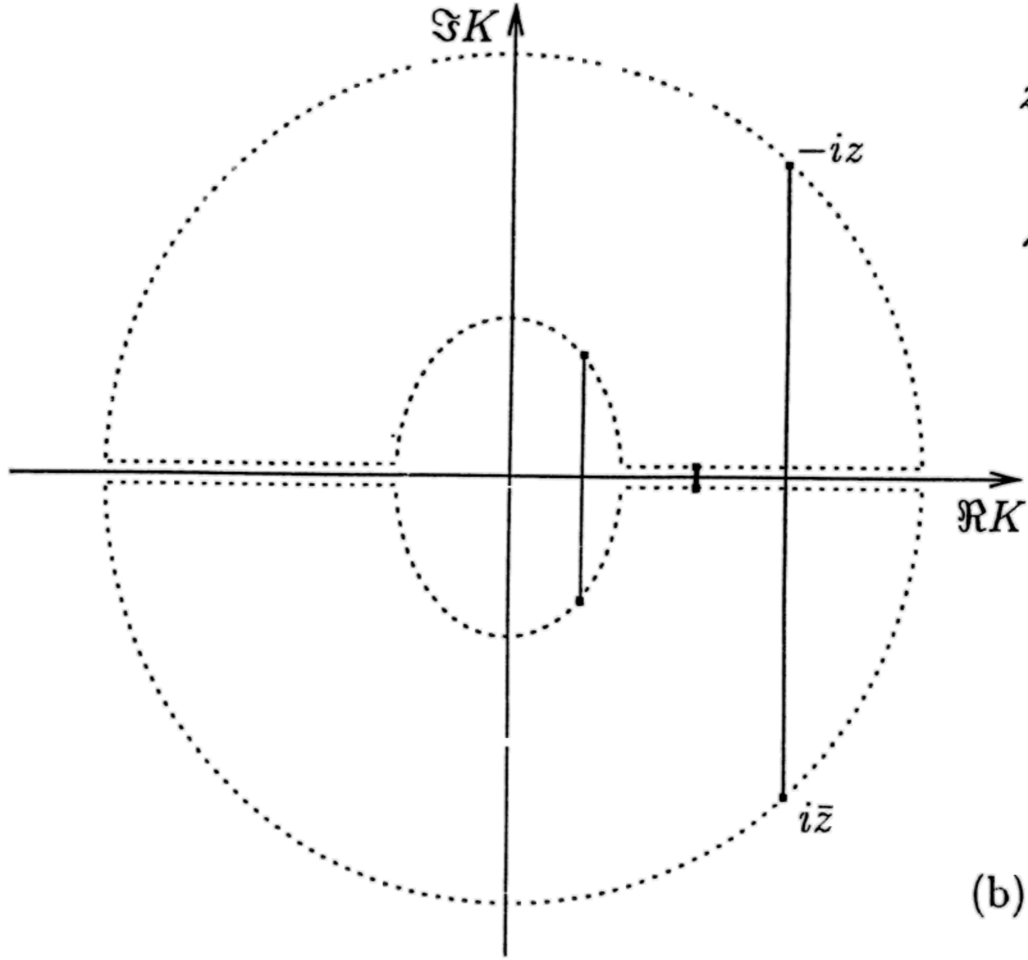
Strategy: ① $\phi(\mathcal{A}^+ - C - \mathcal{A}^-)$: General solution

② $\phi(\mathcal{B})$: Particular solutions (this talk: BH, double BH, disk of dust)

3. Application of the IM to rotating bodies

(for details see Neugebauer, G., Rotating bodies as boundary value problems, Ann. Phys. (Leipzig) 9 (2000) 3-5, 342-354)





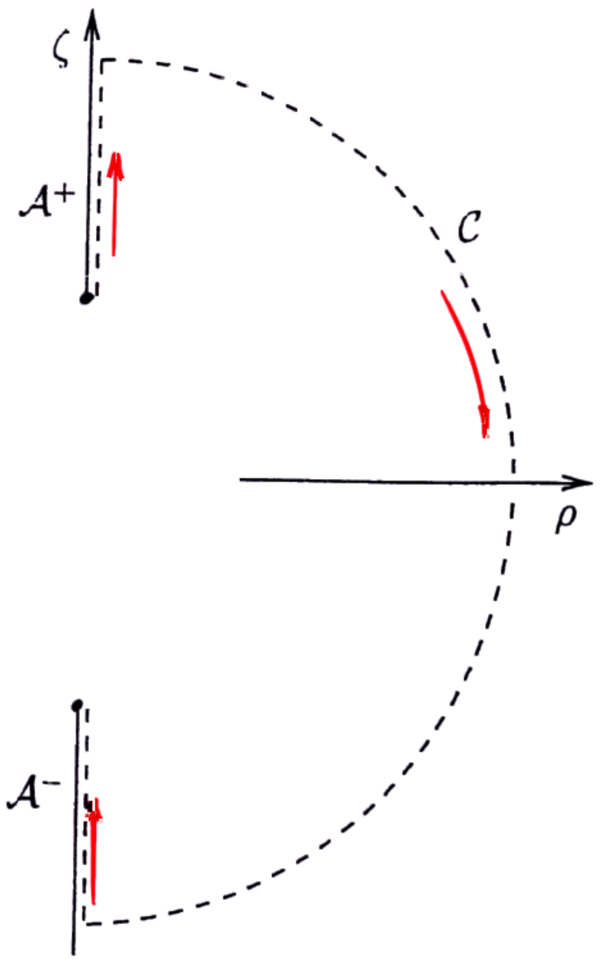
$$z = \rho + i\zeta$$

$$\lambda = \sqrt{\frac{K - i\bar{z}}{K + iz}}$$

Spacetime slice
(t=constant, φ=constant)

Riemann surface
(consisting of 2 K-sheets)

Calculation of $\Phi_{\mathcal{A}^\pm}$ (see "Strategy")



(a) Integration of the LP along $\mathcal{A}^+$-C-$\mathcal{A}^-$ to get ϕ there:

$$\mathcal{A}^+: \quad \phi = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & 1 \end{pmatrix} \begin{pmatrix} F(K) & 0 \\ B(K) & 1 \end{pmatrix} =: \phi_{\mathcal{A}^+} \quad (\zeta \in \mathcal{A}^+)$$

$$C: \quad \phi = \phi_0(K)$$

$$\mathcal{A}^-: \quad \phi = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} 1 & B(K) \\ 0 & F(K) \end{pmatrix} =: \phi_{\mathcal{A}^-} \quad (\zeta \in \mathcal{A}^-)$$

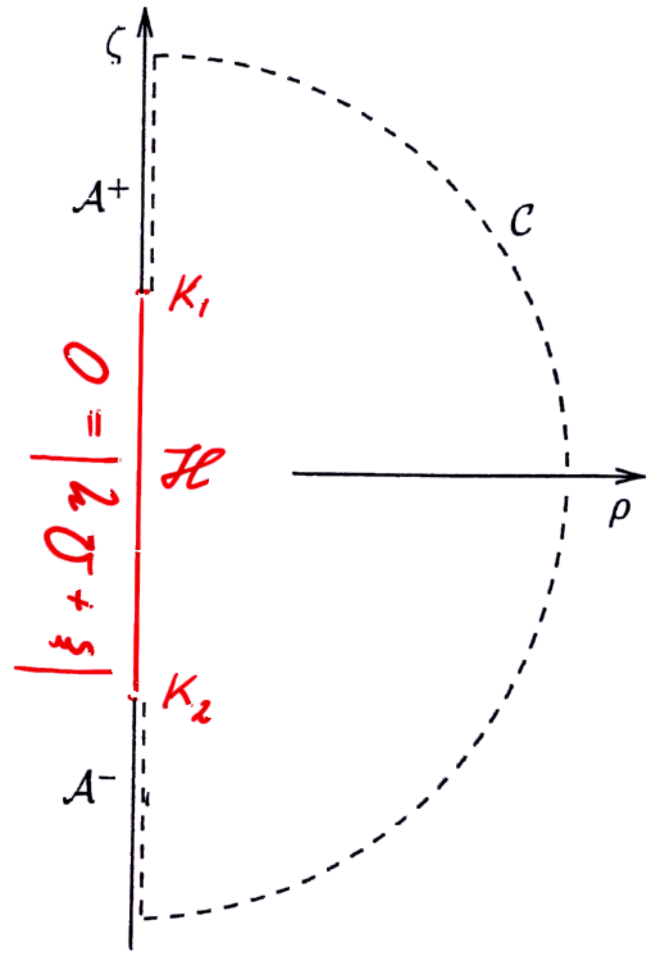
(b) Uniqueness of ϕ in the branch points $K = \zeta$ $(K = i\bar{z},\ K = -iz\ ;\ z = i\zeta,\ \zeta \in \mathcal{A}^\pm)$:

$$\mathcal{A}^+: \ F(\zeta) = \frac{2}{f(\zeta) + \overline{f(\zeta)}}\ , \quad B(\zeta) = \frac{f(\zeta) - \overline{f(\zeta)}}{f(\zeta) + \overline{f(\zeta)}}$$

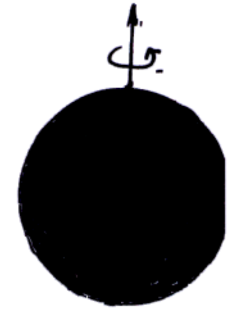
Analytic extension: $F(K) = F(\zeta \to K)$, $B(K) = \ldots$

$$\boxed{\mathcal{A}^\pm: \quad \Phi_{\mathcal{A}^\pm} \longleftrightarrow f(\zeta),\ \zeta \in \mathcal{A}^\pm}$$

3.1 Black hole



$\mathcal{B} = \mathcal{H}$: ① at $\varrho = 0$ ② null KV $\xi + \Omega\eta$
(Carter 1973)



(a) Result of the integration of LP along $\mathcal{H}$:

$$\mathcal{H}: \quad \phi = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & 1 \end{pmatrix} \begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix} =: \phi_{\mathcal{H}}$$

$$\phi' = \mathcal{L}\phi \curvearrowright \phi'_{\mathcal{H}} \quad (\phi' = \phi \text{ in co-rotating frame!})$$

(b) Fieldeqs. hold in K_1, K_2, too:

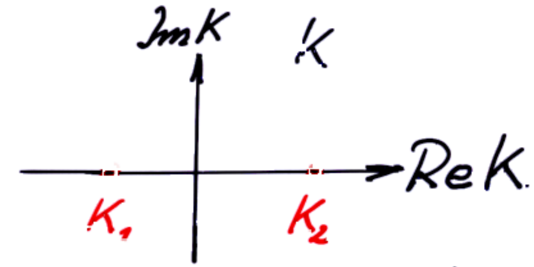
$$\phi_{\mathcal{H}}(K_1) = \phi_{A^+}(K_1), \quad \phi'_{\mathcal{H}}(K_1) = \phi'_{A^+}(K_1);$$

$$\phi_{\mathcal{H}}(K_2) = \phi_{A^-}(K_2), \quad \phi'_{\mathcal{H}}(K_2) = \phi'_{A^-}(K_2)$$

(next transp.)

Notation: $f_i = f(\zeta = K_i)$, Ω: constant angular velocity of BH

$$\mathcal{N} := \begin{pmatrix} F(K) & -B(K) \\ B(K) & \frac{1-B^2}{F} \end{pmatrix}, \quad \mathbb{F}_i = \begin{pmatrix} -f_i & 1 \\ -f_i^2 & f_i \end{pmatrix} \quad (i = 1,2)$$



$$\mathcal{N} = \left(\mathbb{1} + \frac{1}{2i\Omega(K-K_1)}\mathbb{F}_1\right)\left(\mathbb{1} + \frac{1}{2i\Omega(K-K_2)}\mathbb{F}_2\right)$$

(2 simple poles!)

read off: $^{(a)}F(K)$, $B(k) \rightarrow$

$$\mathcal{A}^+(\zeta > K_1): f(\zeta) = \frac{(\zeta + K_1)(1+f_1^2) - 2(1-f_1)}{(\zeta + K_1)(1+f_1^2) - 2f_1(1-f_1)}$$

(w.l.g.: $K_1 = -K_1$, $f_1 = -f_2$)

consequence: $f(\zeta) \rightarrow \phi_{\mathcal{A}^+} \rightarrow \phi(\lambda, \rho, \zeta) \rightarrow f(\rho, \zeta) = \phi_{21}(\lambda = 1, \rho, \zeta)$:

$$\boxed{f(\rho, \zeta) = \frac{r_1 e^{-i\chi} + r_2 e^{i\chi} - 2m\cos\chi}{r_1 e^{-i\chi} + r_2 e^{i\chi} + 2m\cos\chi}}$$

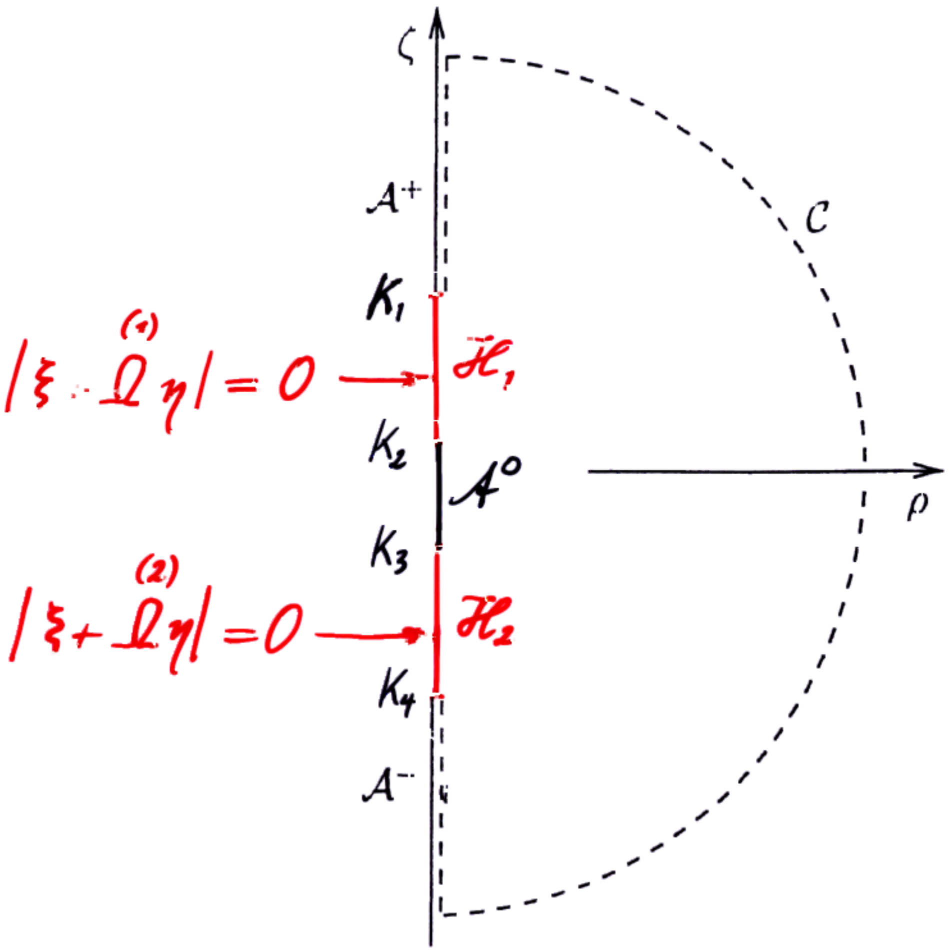
"Kerr solution in Weyl c."

$K_1 = M\cos\chi$, $J/M = \sin\chi$ $\rightarrow \boxed{ds^2}$

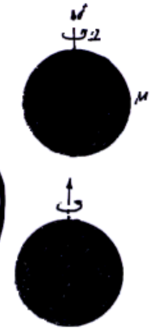
$r_i^2 = (K_i - \zeta)^2 + \rho^2$

RESULT: Axisymm. stationary black hole solution constructed from a BV problem

3.2 Two aligned black holes



$$\mathcal{N} = \prod_{j=1}^{4} \left(\mathbb{1} + \frac{1}{2i\Omega_j(K-K_j)} \mathbb{F}_j \right)^{1)}, \quad \mathcal{N} = \begin{pmatrix} F & -B \\ B & \frac{1-B^2}{F} \end{pmatrix}$$



(a) read off $F(K), B(K)$ to obtain

$\mathcal{A}^+: f(\zeta) \rightarrow \mathcal{E}_{\mathcal{A}^+} \rightarrow \phi(\lambda, \rho, \zeta) \rightarrow f(\rho, \zeta) \rightarrow ds^2$

(b) discuss the 'constraints' $\mathcal{N}_{12} = -\mathcal{N}_{21}$

Ad (a): Theorem: $f(\zeta)$ is a quotient of polynomials $\longleftrightarrow f(\rho,\zeta)$ can be generated by Bäcklund transformations from the Minkowski sp.

The balance problem for Bäcklund-generated solutions was investigated by Kramer & N. (1980), Dietz & Hoenselaers (1983, 85), Tomimatsu & Kihara (1982)

Ad (b): $\rightarrow$ System of non-linear algebraic eqs.

till now: solution of the symmetric case only

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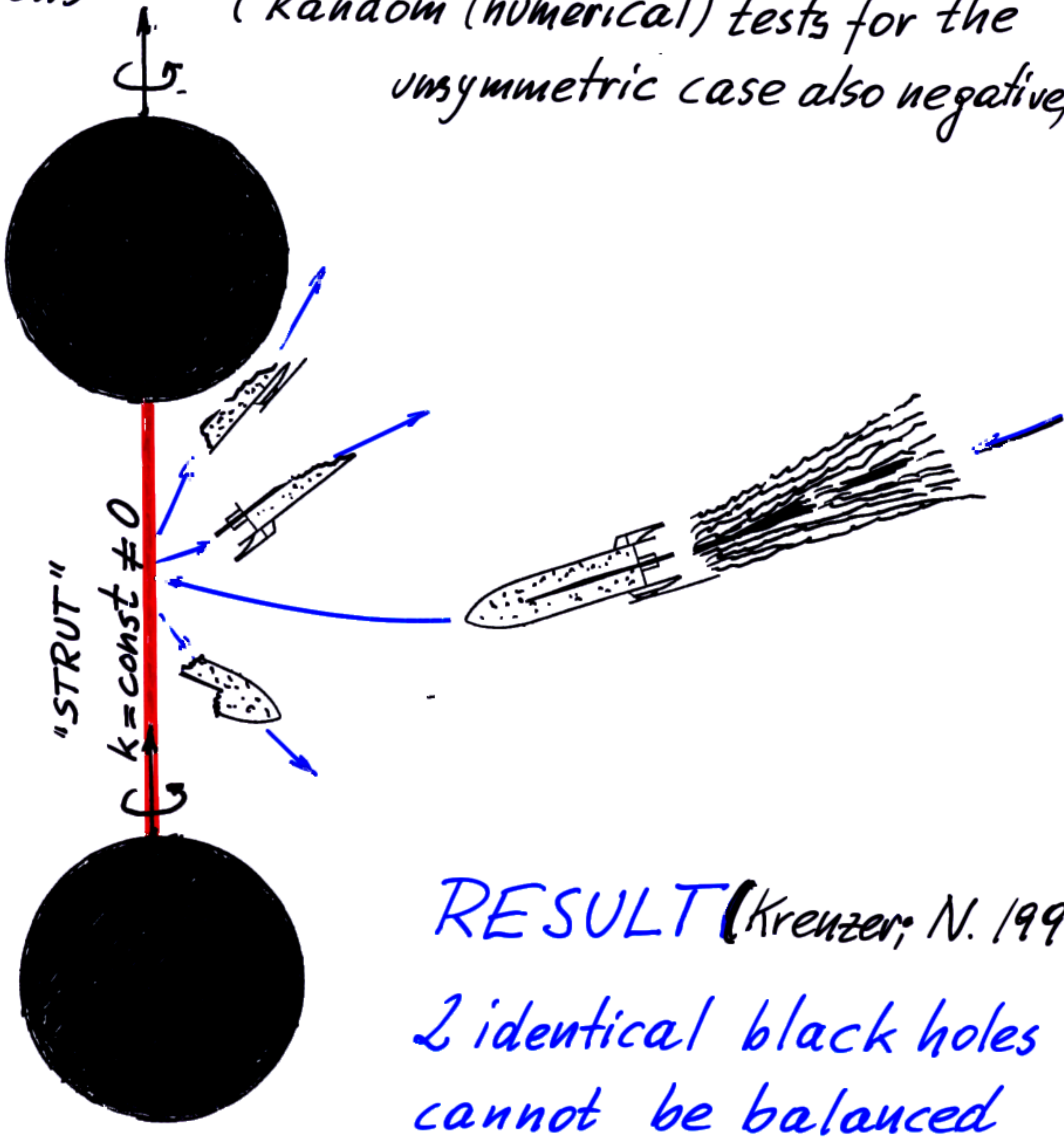
1) $\Omega_1 = \Omega_2 = \overset{(1)}{\Omega}$

$\Omega_3 = \Omega_4 = \overset{(2)}{\Omega}$

Result of the discussion (Dietz & Hoenselaers; Tomimatsu, Kihara)

"The solution for the symmetric balance problem[1] is not in the class of the Bäcklund generated solutions" (Random (numerical) tests for the unsymmetric case also negative)



RESULT (Krenzer; N. 1999)

2 identical black holes cannot be balanced

(cf. Neugebauer, G., Rotating bodies as boundary value problems, Ann. Phys. (Leipzig) 9 (2000) 3-5, 342-354)

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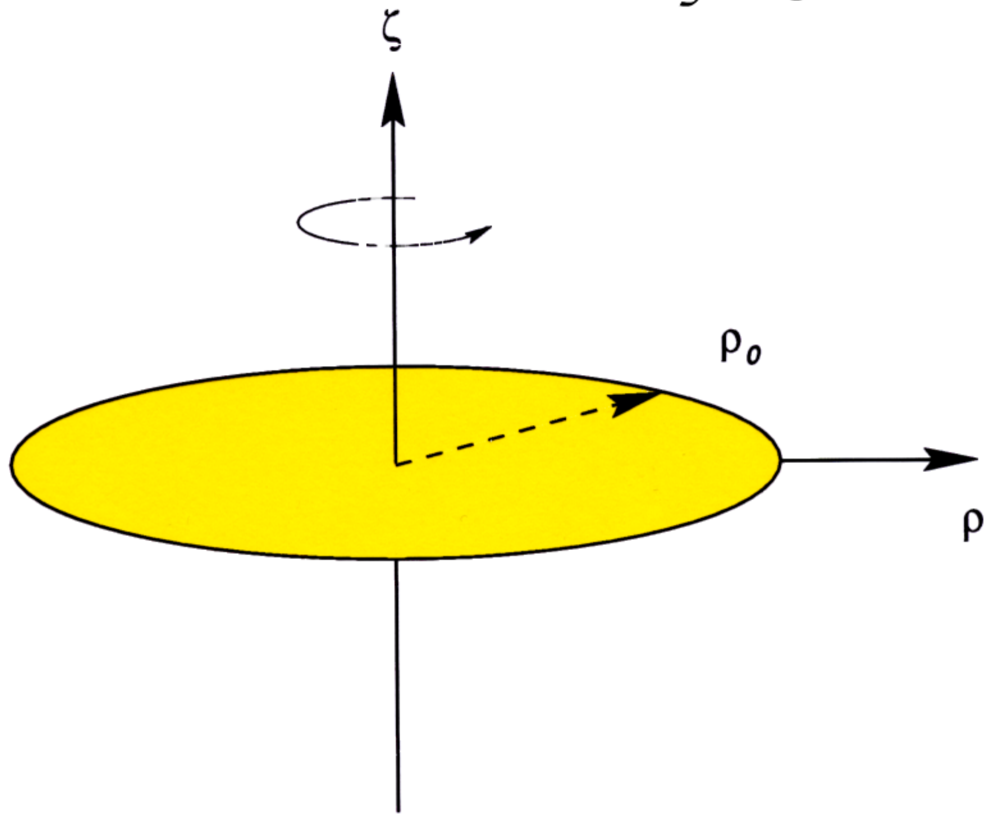
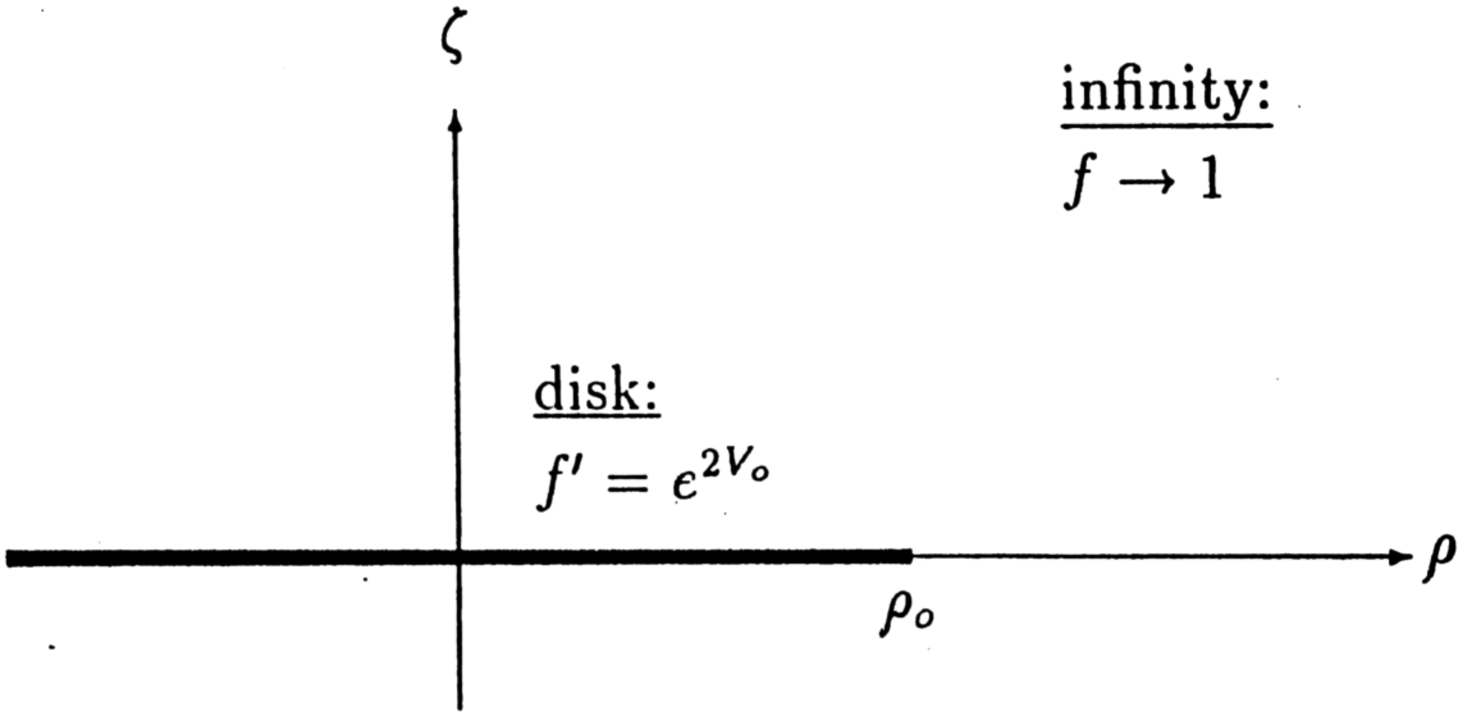
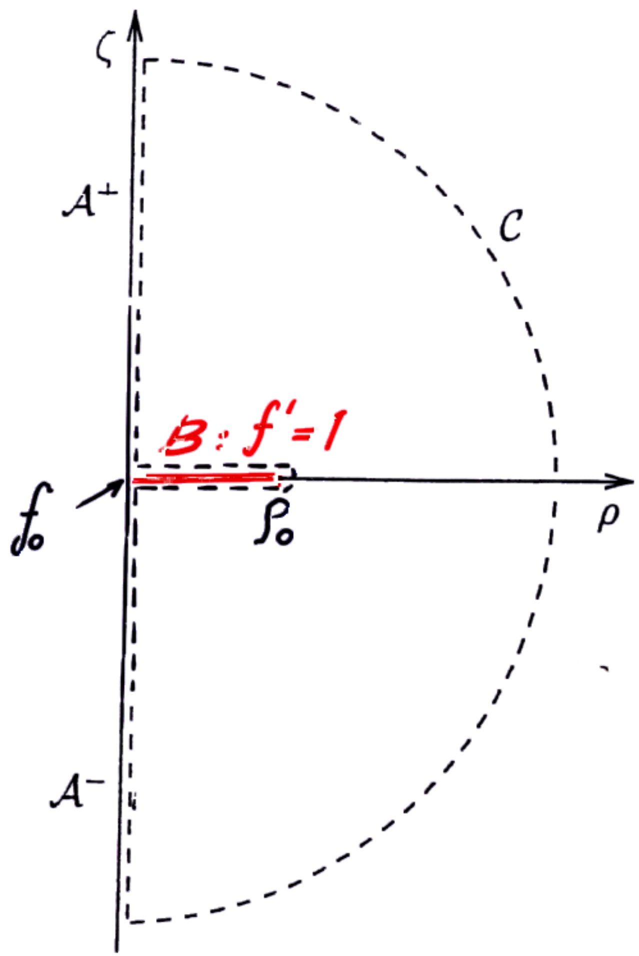
[1] $M_1 = M_2,\ J_1 = J_2,\ \overset{(1)}{\Omega} = \overset{(2)}{\Omega}$

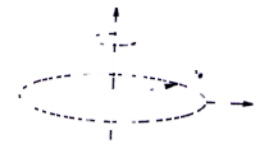
3.3 Disk of Dust

(p=0 limit of perfect fluids, for details see
Neugebauer, G. and Meinel, R., Astroph. J. 414 (1993) L97,
Phys. Rev. Lett. 73 (1994) 2166,
Phys. Rev. Lett. 75 (1995) 3046)

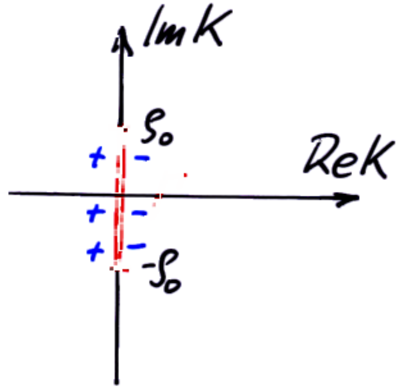








B: Discussion of the LP ↝



$K \in \Gamma$: $S\mathcal{N}_-' = -\mathcal{N}_+ S$

$K \notin \Gamma$: $\mathcal{N}(K)$ analytic in K

"Riemann-Hilbert problem"

$S := P_2(K; \Omega, f_0)$

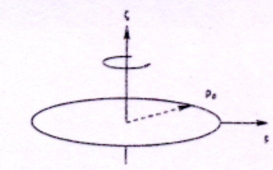
RH-problem explicitely solved

(N. & Meinel 1993, 1995)

$\rightarrow \phi(\lambda, \varrho, \zeta) \longrightarrow f(\varrho, \zeta) \rightarrow ds^2$

RESULT: Solution of the Bardeen-Wagoner problem (1969/71: approximate solution): particles interacting via gravitational forces alone.

2 parameter solution (M, J)

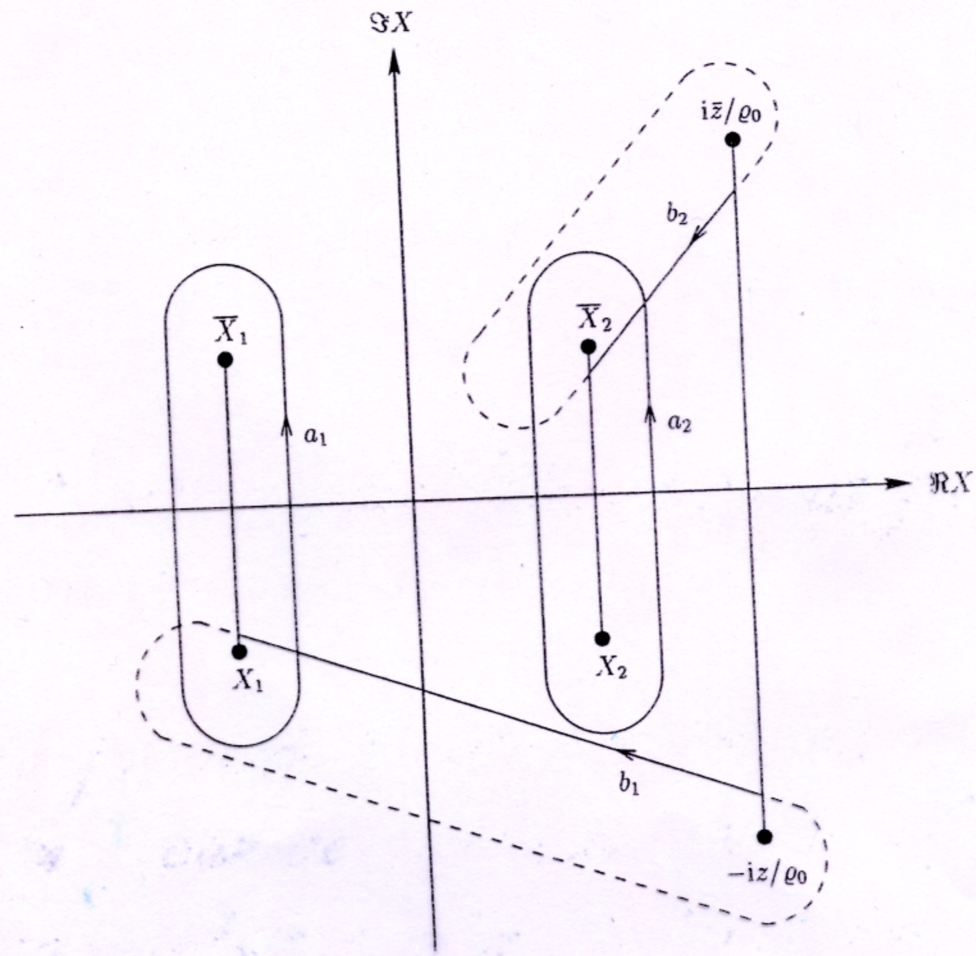


The disk of dust solution (Ω=constant)

Solution in terms of Theta-Functions
(Neugebauer, G., Kleinwächter, A. and Meinel, R., Helv. Phys Acta 69 (1996) 472)

$$\vartheta(x, y; p, q, \alpha) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} p^{m^2} q^{n^2} e^{2mx+2ny+4mn\alpha}$$

$$f = \frac{\vartheta(\alpha_0 u + \alpha_1 v - C_1, \beta_0 u + \beta_1 v - C_2; p, q, \alpha)}{\vartheta(\alpha_0 u + \alpha_1 v + C_1, \beta_0 u + \beta_1 v + C_2; p, q, \alpha)} \, e^{-(\gamma_0 u + \gamma_1 v + \mu w)}$$



$$f(\rho, \zeta) \longrightarrow ds^2$$

SOLUTION IN TERMS OF HYPERELLIPTIC INTEGRALS

$$f(\varrho,\zeta) = \exp\left\{\mu\left[\int_{X_1}^{X_a}\frac{X^2\mathrm{d}X}{W} + \int_{X_2}^{X_b}\frac{X^2\mathrm{d}X}{W} - w\right]\right\}$$

$\mu = 2\Omega_0^2 \varrho_0^2 e^{-2V_0}$

$X = \frac{K}{\varrho_0}$

$$\begin{matrix} X_a(u,v) \\ \Leftarrow \\ X_b(u,v) \end{matrix} \quad \boxed{\begin{aligned} \int_{X_1}^{X_a}\frac{\mathrm{d}X}{W} + \int_{X_2}^{X_b}\frac{\mathrm{d}X}{W} &= u \\ \int_{X_1}^{X_a}\frac{X\mathrm{d}X}{W} + \int_{X_2}^{X_b}\frac{X\mathrm{d}X}{W} &= v \end{aligned}}$$

$$\begin{aligned} W = W_1W_2, \quad W_1 &= \sqrt{\left(X - \frac{\zeta}{\varrho_0}\right)^2 + \left(\frac{\varrho}{\varrho_0}\right)^2}, \\ W_2 &= \sqrt{1 + \mu^2(1 + X^2)^2}, \end{aligned}$$

$$h = \frac{\ln\left(\sqrt{1 + \mu^2(1 + X^2)^2} + \mu(1 + X^2)\right)}{\pi\mathrm{i}\sqrt{1 + \mu^2(1 + X^2)^2}},$$

$$X_1{}^2 = \frac{\mathrm{i} - \mu}{\mu}, \quad X_2{}^2 = -\frac{\mathrm{i} + \mu}{\mu} \quad (\Re X_1 < 0,\ \Re X_2 > 0).$$

$$u = \int_{-\mathrm{i}}^{\mathrm{i}}\frac{h\,\mathrm{d}X}{W_1}, \quad v = \int_{-\mathrm{i}}^{\mathrm{i}}\frac{hX\,\mathrm{d}X}{W_1}, \quad w = \int_{-\mathrm{i}}^{\mathrm{i}}\frac{hX^2\,\mathrm{d}X}{W_1}.$$

$\Delta u = 0$ $\quad$ $\Delta v = 0$ $\quad$ $\Delta w = 0$

$f(\varrho,\zeta) \longrightarrow ds^2 = g_{ik}\,dx^i dx^k$ (VIA INTEGRALS)

Properties of the disk of dust solution

- Dust particles interact via gravition
- No non-gravitational interaction ↘
- geodesic motion of the particles ↘
- "galaxy model (stars = particles)" or
- $p = 0$ limit of a perfect fluid ball

J.M. Bardeen & R.V. Wagoner
Astrophys. J. 158 (1969) L65
Astrophys. J. 167 (1971) 359

G. Neugebauer & R. Meinel
Astrophys. J. 414 (1993) L97
Phys. Rev. Lett 75 (1995) 3046

G. Neugebauer, A. Kleinwächter and R. Meinel,
Helv. Phys. Acta 69 (1996) 472

M, J

- 2-parameter solution: Ω_0, $\mu = 2\Omega_0^2 e^{-2V_0} \varrho_0^2$

  $e^{-V_0} = 1 + z_0$, $\quad 0 \le \mu \le \mu_0 = 4.62966\ldots$

- $\lim\limits_{\mu \to 0}$: Maclaurin disk of Newtonian gravity

- $\lim\limits_{\mu \to \mu_0}$: extreme Kerr BH $\oplus$ ...

- regular everywhere, except disk (continuous!)
- no "$\frac{9}{8} \cdot 2M$ limit"! – Stability?

Special physical properties; Black Hole transition

simple galaxy model

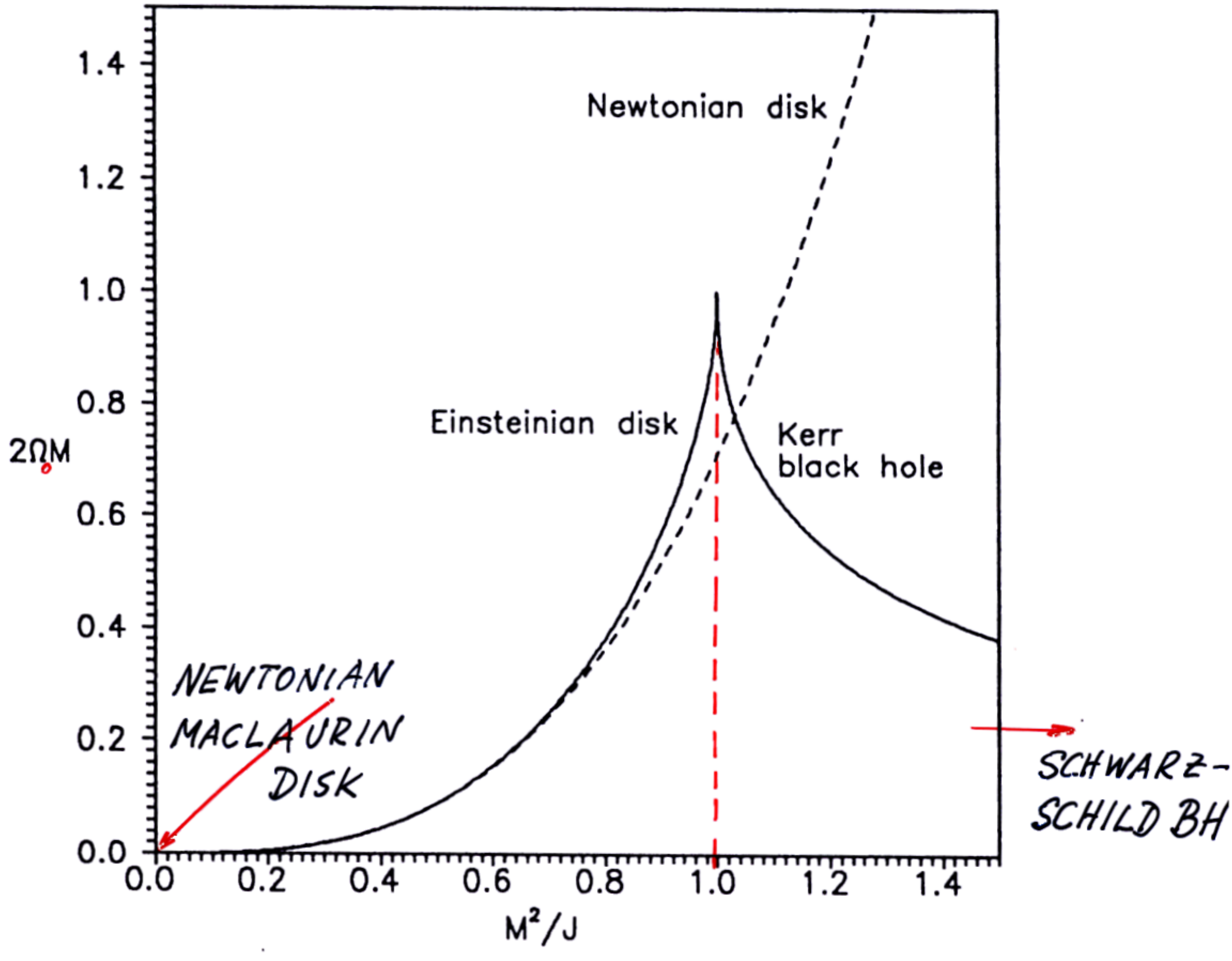
$\mu = 3$: POSITION OF THE OBSERVER:

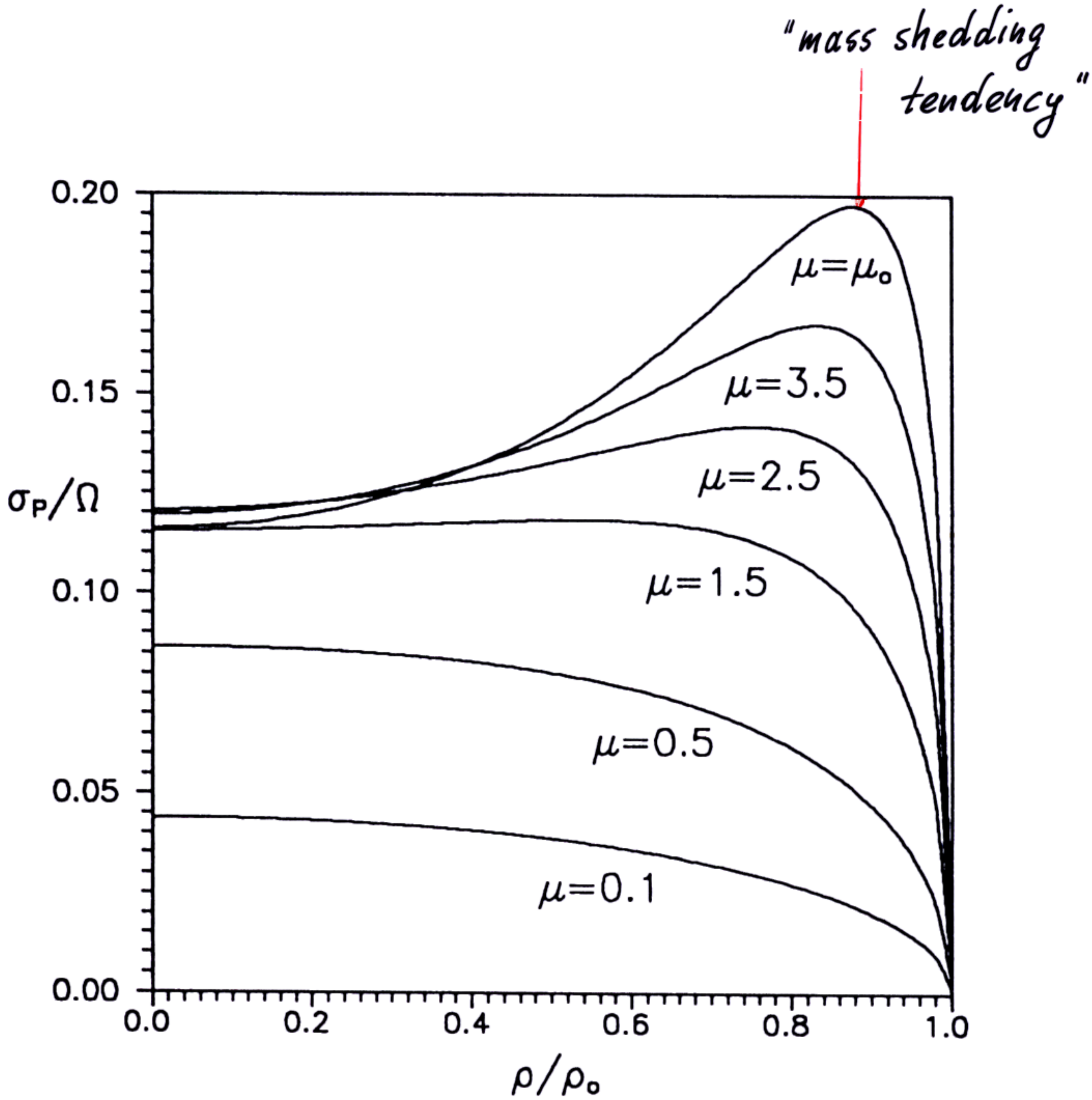
$\varrho = 40\,\varrho_0$

$\zeta = 10\,\varrho_0$

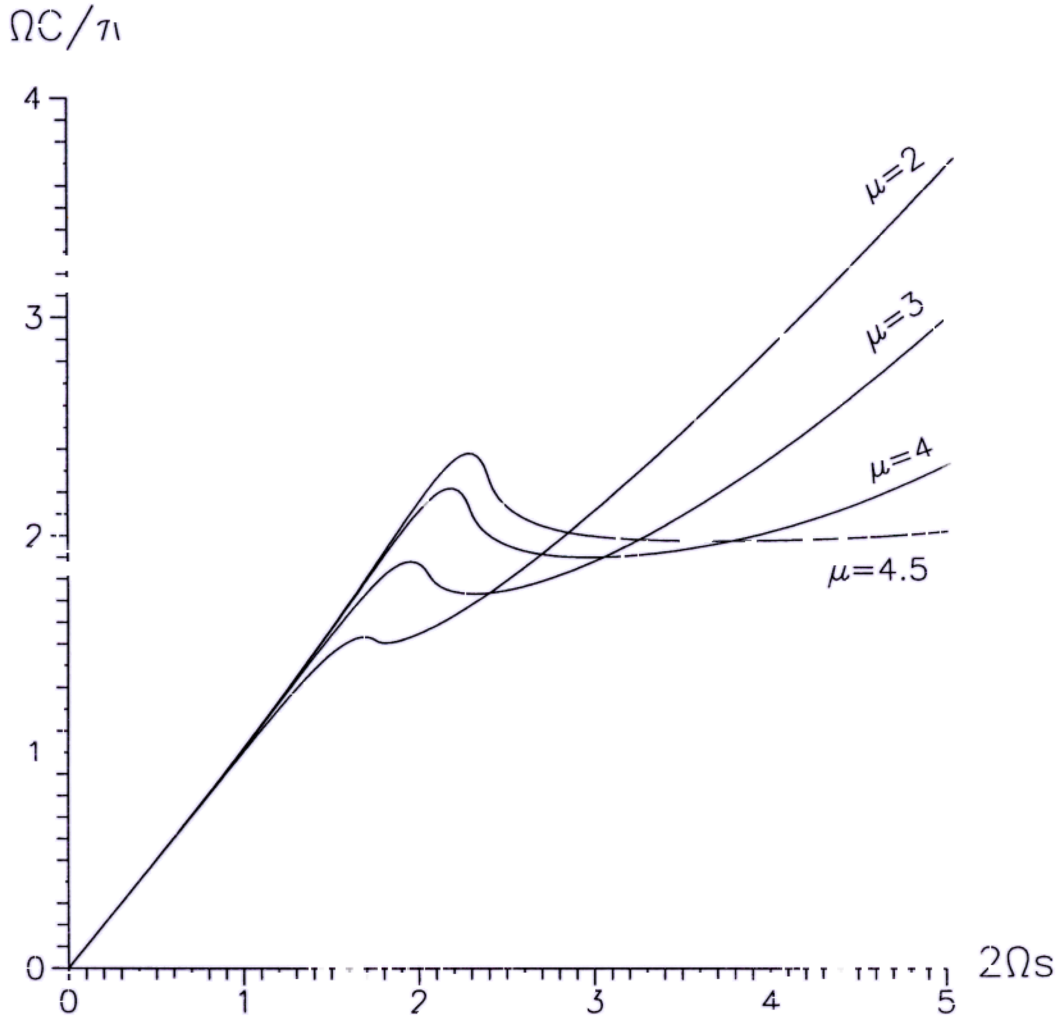
4D ray-tracing picture of the rigidly rotating disk of dust

Phase transition disk of dust / Kerr black hole



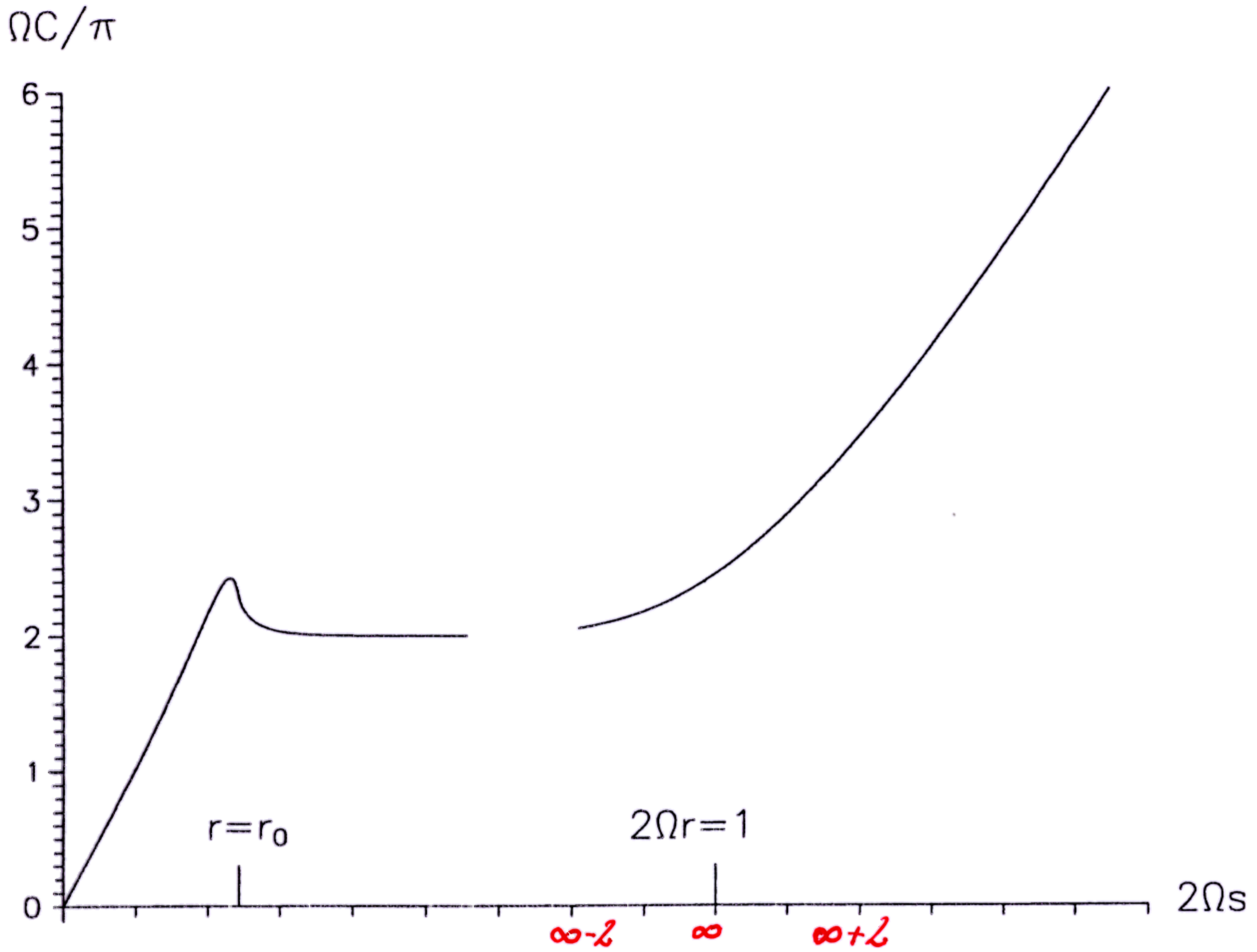


The surface mass–density σ_p



Parametric collapse ($\mu \to \mu_0 = 4.62966\ldots$) of the disk of dust: Shown is the circumference of a circle in the disk plane as a function of the proper distance

Result: 2 limits of spacetime
("disk world" separates from "exterior world" which becomes more and more Kerr-like)



→ In the limit $\mu = \mu_0$ the "disk world" (left branch) and the "world of the extreme Kerr black hole" (right branch) are separated from each other. The point labelled ∞ on the abscissa corresponds to a coordinate radius $r = 1/2\,\Omega$. Points of the "Kerr world" (right branch) are at infinite distance from the disk