Cheeger-Gromov Theory and Applications to General Relativity

Michael T. Anderson

§1. Background: Examples and Definitions.

$\mathbf{M}$ = space of Riemannian metrics on a given manifold M,

a highly non-compact space.

Two simple sources of non-compactness:

• *Diffeomorphisms.* $\mathcal{D}$ = Diff(M), a non-compact group; acts properly on $\mathbf{M}$: if $g \in \mathbf{M}$, $\phi_i \in \mathcal{D}$,

$$\phi_i \to \infty \text{ in } \mathcal{D},$$

then

$$g_i = \phi_i^* g \to \infty \text{ in } \mathbf{M}.$$

All metrics g_i isometric — indistinguishable metrically.
Locally g_i just different local coordinate reps of fixed metric g.

Consider then moduli space

$$\mathcal{M} = \mathbf{M}/\mathcal{D}.$$

• *Scaling.* Given $g \in \mathbf{M}$, $\lambda > 0$, let

$$g_\lambda = \lambda^2 \cdot g.$$

Distances all rescaled by factor of λ. Suppose M compact. If

$$\lambda \to \infty,$$

then (M, g_λ) becomes arbitrarily large:

$$vol_{g_\lambda} M \to \infty, \quad diam_{g_\lambda} M \to \infty, \text{ etc}.$$

No limit metric on M.

Similarly, if

$$\lambda \to 0,$$

then (M, g_λ) becomes arbitrarily small: $(M, g_\lambda) \to \{pt\}$ as metric spaces. Again, no limiting metric on M.

Can combine two divergent behaviors to obtain convergence.

Set $g_\lambda = \lambda^2 \cdot g$ and suppose $\lambda \to \infty$. Fix $p \in M$, and for any fixed $k > 0$, let

$$B_p = B_p(k/\lambda) = \text{geodesic ball}\,.$$

Observe

$$\text{radius} = \begin{cases} k/\lambda \to 0 \text{ in } g \text{ metric} \\ k \text{ in } g_\lambda \text{ metric} \end{cases}$$

Choose chart $\mathcal{U} = \{u_\alpha\}$ for B_p, $u_\alpha(p) = 0$. Define

$$\phi_\lambda(x) = \lambda x.$$

Divergent family of diffeomorphisms. Set

$$u_\alpha^\lambda = \lambda u_\alpha = \phi_\lambda \circ u_\alpha.$$

Then have

$$g_\lambda(\partial/\partial u_\alpha^\lambda, \partial/\partial u_\beta^\lambda) = g(\partial/\partial u_\alpha, \partial/\partial u_\beta) = g_{\alpha\beta}.$$

As $\lambda \to \infty$, $B_p \to p$ and

$$g_{\alpha\beta} \to g_{\alpha\beta}(p) = const.$$

But metrics g_λ defined on their intrinsic geodesic ball of radius k. Thus,

$$(M, \phi_\lambda^*(g_\lambda)) \to (T_p(M), g_0),$$

the flat metric on $T_p(M)$ induced by $g_p = g|_{T_p(M)}$.

"blowing up" process:

restrict to smaller and smaller balls, and blow up to definite size.

The part of M at a definite g-distance to p escapes to infinity: not detected in the limit g_0. Thus,

attach base points to the blow-ups;

different base points may give rise to different limits, (although here all pointed limits are isometric).

Penrose Limits: analogous blow-up process for Lorentz metrics; non-isotropic blow-up in directions orthogonal to null geodesic. Limits are plane gravitational waves — non-flat.

If (M, g) complete and non-compact, do "blow-down", with

$$\lambda \to 0:$$

geodesic balls, of large radius $B_p(k/\lambda)$ rescaled down to fixed size k.

This useful in understanding the large scale or asymptotic behavior of the metric.

Definition: Convergence of Metrics. Let $L^{k,p} = L^{k,p}(\Omega) =$ Sobolev space: k weak derivatives in L^p, $\Omega \subset \mathbb{R}^n$.

A sequence of metrics g_i on manifolds M_i converges in $L^{k,p}$ topology to limit metric g on manifold M if $\exists$ charts $\Phi = \{\phi_k\}$, covering M, and diffeomorphisms $F_i : M \to M_i$, s.t.

$$(F_i^* g_i)_{\alpha\beta} \to g_{\alpha\beta}, \tag{1.1}$$

in the $L^{k,p}$ topology.

Same definition holds for convergence in the $C^{k,\alpha}$ topology, as well as the weak $L^{k,p}$ topology.

(Recall $f_i \to f$ weakly in $L^p(\Omega)$ iff $\int f_i g \to \int fg$, $\forall g \in L^q(\Omega)$, $p^{-1} + q^{-1} = 1$.)

To obtain local control on a metric, or sequence of metrics, assume

curvature bounds.

Cheeger-Gromov theory requires a bound on the full Riemann curvature tensor

$$|Riem| \leq K, \tag{1.2}$$

for some $K < \infty$. Since {# components of Riem} >> {# components of metric}, (in dimensions ≥ 4), this is overdetermined set of constraints on the metric and so overly restrictive.

Much more natural to impose bounds on the Ricci curvature

$$|Ric| \leq k. \tag{1.3}$$

Of course, bounds on Ricci natural in general relativity, via the Einstein equations. So emphasize (1.3) over (1.2) whenever possible.

Can view Cheeger-Gromov theory as generalization of Teichmüller theory to higher dimensions and variable curvature.

Teichmüller theory describes the moduli space $\mathcal{M}_c$ of constant curvature metrics on surfaces. On closed surfaces, have a

basic trichotomy

for behavior of sequences in $\mathcal{M}_c$, normalized to unit area:

- *Compactness/Convergence*. A sequence $g_i \in \mathcal{M}_c$ has a subsequence converging to a limit $g \in \mathcal{M}_c$. Convergence is understood to be modulo diffeomorphisms. For instance this is always the case on S^2, since $\mathcal{M}_c = \{pt\}$ on S^2.

- *Collapse*. The sequence $g_i \in \mathcal{M}_c$ collapses everywhere, in that

  $$inj_{g_i}(x) \to 0$$

  at every x. Occurs only on torus T^2. Metrics become very long and very thin.

  No limit metric on T^2. Choosing base points x_i, consider based sequences (T^2, g_i, x_i); limits are then the collapsed space $(\mathbb{R}, g_\infty, x_\infty)$.

- *Cusp Formation.* Mixture of the two previous cases. Occurs only for hyperbolic metrics, i.e. on surfaces Σ_g of genus $g \geq 2$.

Some based sequences $(\Sigma_g, g_i, x_i) \to$ limit $(\Sigma, g_\infty, x_\infty) =$ complete non-compact hyperbolic surface of finite volume. Ends collapse.

Other based sequences (Σ, g_i, y_i) collapse — long thin annuli collapse to $\mathbb{R}$.

§2. Convergence/Compactness.

Issue: (Pre)-compactness of a family of metrics on manifold M.

Main Point. Establish a lower bound on the radius of balls on which have apriori control of the metric in a given topology —$C^{k,\alpha}$ or $L^{k,p}$.

Given such uniform local control, then usually easy to obtain global control, via suitable global assumptions on size of M: volume, diameter, etc. — Choice of scale.

To obtain local control, first important step: choose

$$\text{“good gauge”} = \text{local coord. system for } g$$

Look at coordinates built from the geometry of the metric itself.

(i) geodesic normal coordinates

(ii) distance coordinates

Entail loss of derivatives: 2 or 1 resp.

Optimal choice:

(iii) harmonic coordinates

Definition: harmonic radius r_h. Fix a function topology, say $L^{k,p}$ and a constant $C > 1$. Given $x \in (M, g)$, the $L^{k,p}$ harmonic radius

$$r_h(x) = r_h^{k,p}(x)$$

= largest radius s.t. on ball $B_x(r_h(x))$ have harmonic coordinate chart $U = \{u_\alpha\}$ in which the metric $g = g_{\alpha\beta}$ is controlled in $L^{k,p}$ norm: thus,

$$C^{-1}\delta_{\alpha\beta} \leq g_{\alpha\beta} \leq C\delta_{\alpha\beta}, \quad \text{(as bilinear forms)}, \tag{2.1}$$

$$[r_h(x)]^{kp-n} \int_{B_x(r_h(x))} |\partial^k g_{\alpha\beta}|^p dV \leq C. \tag{2.2}$$

$kp > n$, so that $L^{k,p} \subset C^0$. Estimates are scale invariant, (rescale harmonic coordinates also), so r_h scales as a distance.

Important fact:

r_h is continuous in (strong) $L^{k,p}$ topology on $\mathbf{M}$.

Similar definition and properties for $C^{k,\alpha}$ harmonic radii.

Suppose g_i is a sequence of metrics on a manifold M with a uniform lower bound on r_h,

$$r_h(x, g_i) \geq r_0 > 0,$$

some fixed r_0.

On each r_0-ball, have then $L^{k,p}$ control of $(g_i)_{\alpha\beta}$ - in harmonic coords.

Banach-Alaoglu theorem: bounded sequences are weakly compact in Banach spaces. So, on each ball,

$$(g_i)_{\alpha\beta} \to (g_\infty)_{\alpha\beta}$$

weakly in $L^{2,p}$.

Easy to see that harmonic charts for $g_i \to$ harmonic chart for g_∞ weakly in $L^{k+1,p}$. Also, overlaps of charts are in $L^{k+1,p}$.

Thus get limit $L^{k,p}$ metric g_∞: convergence to limit is in weak $L^{k,p}$ topology.

[Same type of arguments hold w.r.t $C^{k,\alpha}$ topology, via the Arzela-Ascoli theorem; use $C^{k,\alpha'}$, $\alpha' < \alpha$, in place of weak $L^{k,p}$.]

Thus, main issue to obtain convergence:

obtain lower bound on r_h

under bounds on geometry. Next result is one typical example.

Theorem 2.1 Convergence I. *Let M be an n-manifold and let $\mathcal{M}(\lambda, i_o, D)$ be the space of Riemannian metrics such that*

$$|Ric| \leq k, \; inj \geq i_o, \; diam \leq D. \tag{2.3}$$

Then $\mathcal{M}(\lambda, i_o, D)$ is precompact in the $C^{1,\alpha}$ and weak $L^{2,p}$ topologies, any $\alpha < 1$ and $p < \infty$.

Sketch of Proof: Prove a uniform lower bound on the $L^{2,p}$ harmonic radius $r_h = r_h^{2,p}$, i.e.

$$r_h(x) \geq r_o = r_o(k, i_o, D), \tag{2.4}$$

under the bounds (2.3).

Overall, prove (2.4) by contradiction. Thus, if (2.4) is false, $\exists\{g_i\}$ on M, satisfying (2.3), but

$$r_h(x_i) \to 0,$$

for some $x_i \in M$. Assume x_i realize the minimal value of r_h on (M, g_i). Then rescale the metrics g_i by this minimal harmonic radius, i.e. set

$$\bar{g}_i = r_h(x_i)^{-2} \cdot g_i. \tag{2.5}$$

For $\bar{r}_h =$ harmonic radius w.r.t. $\bar{g}$, scaling properties give

$$\bar{r}_h(x_i) = 1, \quad \text{and} \quad \bar{r}_h(y_i) \geq 1, \tag{2.6}$$

for all $y_i \in (M, \bar{g}_i)$.

Thus, pointed Riemannian manifolds $(M, \bar{g}_i, x_i)$ have a subsequence converging in the *weak* $L^{2,p}$ topology to a limit $L^{2,p}$ Riemannian manifold $(N, \bar{g}_\infty, x_\infty)$.

So far, nothing essential has been done —the construction above more or less amounts to just renormalizations.

There are two basic ingredients in obtaining further control however, one geometric and one analytic.

Geometric argument

Limit space $(N, \bar{g}_\infty)$ is Ricci-flat: in scale $\bar{g}_i$,

$$|Ric_{\bar{g}_i}| \leq k \cdot r_h(x_i) \to 0, \quad \text{as} \quad i \to \infty. \tag{2.7}$$

Next, injectivity radius in scale $\bar{g}_i$ satisfies

$$inj_{\bar{g}_i} \geq i_o \cdot r_h(x_i)^{-1} \to \infty, \quad \text{as} \quad i \to \infty. \tag{2.8}$$

So roughly, limit $(N, \bar{g}_\infty)$ has $inj_{g_\infty} = \infty$.

In fact, (2.8) implies $(M, \bar{g}_i)$ contains arbitrarily long, (depending on i), minimizing geodesics, centered at x_i, in all directions. So limit $(N, \bar{g}_\infty)$ has line thru x_∞ in all directions.

Splitting theorem (Cheeger-Gromoll):

A complete manifold (N, g), $\text{Ric}_g \geq 0$, splits isometrically along any line.

Hence,

$$(N, \bar{g}_\infty) = (\mathbb{R}^n, g_o)\,.$$

But, of course,

$$r_h(\mathbb{R}^n, g_o) = \infty\,.$$

If the convergence $(M, \bar{g}_i, x_i) \to (\mathbb{R}^n, g_o, x_\infty)$ is in strong $L^{2,p}$ topology, then the continuity of r_h gives contradiction, since have normalized

$$\bar{r}_h(x_i) = 1$$

.

Idea here: use elliptic regularity to bootstrap or improve smoothness of the convergence.

In harmonic coordinates, Ricci curvature has especially simple form:

$$-\tfrac{1}{2}\Delta g_{\alpha\beta} + Q_{\alpha\beta}(g, \partial g) = Ric_{\alpha\beta}, \tag{2.9}$$

where $\Delta = g^{\alpha\beta}\partial_\alpha\partial_\beta$.

If $r_h(x) = 1$ and $r_h(y) \geq r_o > 0$, $\forall y \in \partial B_x(1)$, then have uniform $L^{1,p}$ bound on Q and uniform $L^{2,p}$ bounds on the coefficients for the Laplacian within $B_x(1 + \frac{1}{2}r_o)$.

Suppose Ric is uniformly bounded in L^∞. Then elliptic regularity applied to (2.9) implies that

$$||g_{\alpha\beta}||_{L^{2,q}} \leq C = C(q),$$

for any $q < \infty$, ($q > p$!).

More importantly, if g_i is a sequence of metrics s.t.

$$(Ric_{g_i})_{\alpha\beta} \Rightarrow (Ric_{g_\infty})_{\alpha\beta},$$

strongly in L^p, then elliptic regularity again implies that

$$(g_i)_{\alpha\beta} \Rightarrow (g_\infty)_{\alpha\beta}.$$

This essentially proves that the $L^{2,p}$ harmonic radius is continuous w.r.t. the strong $L^{2,p}$ topology. Further, when applied to the sequence $\bar{g}_i$ — using $\bar{r}_h \geq 1$ everywhere — it implies that the metrics $\bar{g}_i$ converge strongly in $L^{2,p}$ to the limit $\bar{g}_\infty$.