Cheeger-Gromov Theory and Applications to General Relativity

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§1. Background: Examples and Definitions.

$$\mathbf{M} = \text{space of Riemannian metrics on a given manifold } M,$$

a highly non-compact space.

Two simple sources of non-compactness:

- *Diffeomorphisms.* $\mathcal{D} = \text{Diff(M)}$, a non-compact group; acts properly on $\mathbf{M}$: if $g \in \mathbf{M}$, $\phi_i \in \mathcal{D}$,

$$\phi_i \to \infty \text{ in } \mathcal{D},$$

then

$$g_i = \phi_i^* g \to \infty \text{ in } \mathbf{M}.$$

All metrics g_i isometric — indistinguishable metrically.
Locally g_i just different local coordinate reps of fixed metric g.

Consider then moduli space

$$\mathcal{M} = \mathbf{M}/\mathcal{D}.$$

- *Scaling.* Given $g \in \mathbf{M}$, $\lambda > 0$, let

$$g_\lambda = \lambda^2 \cdot g.$$

Distances all rescaled by factor of λ. Suppose M compact. If

$$\lambda \to \infty,$$

then (M, g_λ) becomes arbitrarily large:

$$vol_{g_\lambda} M \to \infty, \quad diam_{g_\lambda} M \to \infty, \text{ etc}.$$

No limit metric on M.

Similarly, if

$$\lambda \to 0,$$

then (M, g_λ) becomes arbitrarily small: $(M, g_\lambda) \to \{pt\}$ as metric spaces. Again, no limiting metric on M.

Can combine two divergent behaviors to obtain convergence.

Set $g_\lambda = \lambda^2 \cdot g$ and suppose $\lambda \to \infty$. Fix $p \in M$, and for any fixed $k > 0$, let

$$B_p = B_p(k/\lambda) = \text{geodesic ball}\,.$$

Observe

$$\text{radius} = \begin{cases} k/\lambda \to 0 \text{ in } g \text{ metric} \\ k \text{ in } g_\lambda \text{ metric} \end{cases}$$

Choose chart $\mathcal{U} = \{u_\alpha\}$ for B_p, $u_\alpha(p) = 0$. Define

$$\phi_\lambda(x) = \lambda x.$$

Divergent family of diffeomorphisms. Set

$$u_\alpha^\lambda = \lambda u_\alpha = \phi_\lambda \circ u_\alpha.$$

Then have

$$g_\lambda(\partial/\partial u_\alpha^\lambda, \partial/\partial u_\beta^\lambda) = g(\partial/\partial u_\alpha, \partial/\partial u_\beta) = g_{\alpha\beta}.$$

As $\lambda \to \infty$, $B_p \to p$ and

$$g_{\alpha\beta} \to g_{\alpha\beta}(p) = const.$$

But metrics g_λ defined on their intrinsic geodesic ball of radius k. Thus,

$$(M, \phi_\lambda^*(g_\lambda)) \to (T_p(M), g_0),$$

the flat metric on $T_p(M)$ induced by $g_p = g|_{T_p(M)}$.

"blowing up" process:

restrict to smaller and smaller balls, and blow up to definite size.

The part of M at a definite g-distance to p escapes to infinity: not detected in the limit g_0. Thus,

attach base points to the blow-ups;

different base points may give rise to different limits, (although here all pointed limits are isometric).

Penrose Limits: analogous blow-up process for Lorentz metrics; non-isotropic blow-up in directions orthogonal to null geodesic. Limits are plane gravitational waves — non-flat.

If (M, g) complete and non-compact, do "blow-down", with

$$\lambda \to 0:$$

geodesic balls, of large radius $B_p(k/\lambda)$ rescaled down to fixed size k.

This useful in understanding the large scale or asymptotic behavior of the metric.

Definition: Convergence of Metrics. Let $L^{k,p} = L^{k,p}(\Omega) =$ Sobolev space: k weak derivatives in L^p, $\Omega \subset \mathbb{R}^n$.

A sequence of metrics g_i on manifolds M_i converges in $L^{k,p}$ topology to limit metric g on manifold M if $\exists$ charts $\Phi = \{\phi_k\}$, covering M, and diffeomorphisms $F_i : M \to M_i$, s.t.

$$(F_i^* g_i)_{\alpha\beta} \to g_{\alpha\beta}, \tag{1.1}$$

in the $L^{k,p}$ topology.

Same definition holds for convergence in the $C^{k,\alpha}$ topology, as well as the weak $L^{k,p}$ topology.

(Recall $f_i \to f$ weakly in $L^p(\Omega)$ iff $\int f_i g \to \int fg$, $\forall g \in L^q(\Omega)$, $p^{-1} + q^{-1} = 1$.)

To obtain local control on a metric, or sequence of metrics, assume

curvature bounds.

Cheeger-Gromov theory requires a bound on the full Riemann curvature tensor

$$|Riem| \leq K, \tag{1.2}$$

for some $K < \infty$. Since {# components of Riem} >> {# components of metric}, (in dimensions ≥ 4), this is overdetermined set of constraints on the metric and so overly restrictive.

Much more natural to impose bounds on the Ricci curvature

$$|Ric| \leq k. \tag{1.3}$$

Of course, bounds on Ricci natural in general relativity, via the Einstein equations. So emphasize (1.3) over (1.2) whenever possible.

Can view Cheeger-Gromov theory as generalization of Teichmüller theory to higher dimensions and variable curvature.

Teichmüller theory describes the moduli space $\mathcal{M}_c$ of constant curvature metrics on surfaces. On closed surfaces, have a

basic trichotomy

for behavior of sequences in $\mathcal{M}_c$, normalized to unit area:

- *Compactness/Convergence.* A sequence $g_i \in \mathcal{M}_c$ has a subsequence converging to a limit $g \in \mathcal{M}_c$. Convergence is understood to be modulo diffeomorphisms. For instance this is always the case on S^2, since $\mathcal{M}_c = \{pt\}$ on S^2.

- *Collapse.* The sequence $g_i \in \mathcal{M}_c$ collapses everywhere, in that

$$inj_{g_i}(x) \to 0$$

at every x. Occurs only on torus T^2. Metrics become very long and very thin.

No limit metric on T^2. Choosing base points x_i, consider based sequences (T^2, g_i, x_i); limits are then the collapsed space $(\mathbb{R}, g_\infty, x_\infty)$.

- *Cusp Formation.* Mixture of the two previous cases. Occurs only for hyperbolic metrics, i.e. on surfaces Σ_g of genus $g \geq 2$.

Some based sequences $(\Sigma_g, g_i, x_i) \to$ limit $(\Sigma, g_\infty, x_\infty) =$ complete non-compact hyperbolic surface of finite volume. Ends collapse.

Other based sequences (Σ, g_i, y_i) collapse — long thin annuli collapse to $\mathbb{R}$.

§2. Convergence/Compactness.

Issue: (Pre)-compactness of a family of metrics on manifold M.

Main Point. Establish a lower bound on the radius of balls on which have apriori control of the metric in a given topology —$C^{k,\alpha}$ or $L^{k,p}$.

Given such uniform local control, then usually easy to obtain global control, via suitable global assumptions on size of M: volume, diameter, etc. — Choice of scale.

To obtain local control, first important step: choose

"good gauge" = local coord. system for g

Look at coordinates built from the geometry of the metric itself.

(i) geodesic normal coordinates

(ii) distance coordinates

Entail loss of derivatives: 2 or 1 resp.

Optimal choice:

(iii) harmonic coordinates

Definition: harmonic radius r_h. Fix a function topology, say $L^{k,p}$ and a constant $C > 1$. Given $x \in (M, g)$, the $L^{k,p}$ harmonic radius

$$r_h(x) = r_h^{k,p}(x)$$

= largest radius s.t. on ball $B_x(r_h(x))$ have harmonic coordinate chart $U = \{u_\alpha\}$ in which the metric $g = g_{\alpha\beta}$ is controlled in $L^{k,p}$ norm: thus,

$$C^{-1}\delta_{\alpha\beta} \leq g_{\alpha\beta} \leq C\delta_{\alpha\beta}, \quad \text{(as bilinear forms)}, \tag{2.1}$$

$$[r_h(x)]^{kp-n} \int_{B_x(r_h(x))} |\partial^k g_{\alpha\beta}|^p dV \leq C. \tag{2.2}$$

$kp > n$, so that $L^{k,p} \subset C^0$. Estimates are scale invariant, (rescale harmonic coordinates also), so r_h scales as a distance.

Important fact:

$$r_h \text{ is continuous in (strong) } L^{k,p} \text{ topology on } \mathbf{M}.$$

Similar definition and properties for $C^{k,\alpha}$ harmonic radii.

Suppose g_i is a sequence of metrics on a manifold M with a uniform lower bound on r_h,

$$r_h(x, g_i) \geq r_0 > 0,$$

some fixed r_0.

On each r_0-ball, have then $L^{k,p}$ control of $(g_i)_{\alpha\beta}$ - in harmonic coords.

Banach-Alaoglu theorem: bounded sequences are weakly compact in Banach spaces. So, on each ball,

$$(g_i)_{\alpha\beta} \to (g_\infty)_{\alpha\beta}$$

weakly in $L^{2,p}$.

Easy to see that harmonic charts for $g_i \to$ harmonic chart for g_∞ weakly in $L^{k+1,p}$. Also, overlaps of charts are in $L^{k+1,p}$.

Thus get limit $L^{k,p}$ metric g_∞: convergence to limit is in weak $L^{k,p}$ topology.

[Same type of arguments hold w.r.t $C^{k,\alpha}$ topology, via the Arzela-Ascoli theorem; use $C^{k,\alpha'}, \alpha' < \alpha$, in place of weak $L^{k,p}$.]

Thus, main issue to obtain convergence:

$$\text{obtain lower bound on } r_h$$

under bounds on geometry. Next result is one typical example.

Theorem 2.1 Convergence I. *Let M be an n-manifold and let $\mathcal{M}(\lambda, i_o, D)$ be the space of Riemannian metrics such that*

$$|Ric| \leq k, \ inj \geq i_o, \ diam \leq D. \tag{2.3}$$

Then $\mathcal{M}(\lambda, i_o, D)$ is precompact in the $C^{1,\alpha}$ and weak $L^{2,p}$ topologies, any $\alpha < 1$ and $p < \infty$.

Sketch of Proof: Prove a uniform lower bound on the $L^{2,p}$ harmonic radius $r_h = r_h^{2,p}$, i.e.

$$r_h(x) \geq r_o = r_o(k, i_o, D), \tag{2.4}$$

under the bounds (2.3).

Overall, prove (2.4) by contradiction. Thus, if (2.4) is false, $\exists \{g_i\}$ on M, satisfying (2.3), but

$$r_h(x_i) \to 0,$$

for some $x_i \in M$. Assume x_i realize the minimal value of r_h on (M, g_i). Then rescale the metrics g_i by this minimal harmonic radius, i.e. set

$$\bar{g}_i = r_h(x_i)^{-2} \cdot g_i. \tag{2.5}$$

For $\bar{r}_h =$ harmonic radius w.r.t. $\bar{g}$, scaling properties give

$$\bar{r}_h(x_i) = 1, \quad \text{and} \quad \bar{r}_h(y_i) \geq 1, \tag{2.6}$$

for all $y_i \in (M, \bar{g}_i)$.

Thus, pointed Riemannian manifolds $(M, \bar{g}_i, x_i)$ have a subsequence converging in the *weak* $L^{2,p}$ topology to a limit $L^{2,p}$ Riemannian manifold $(N, \bar{g}_\infty, x_\infty)$.

So far, nothing essential has been done —the construction above more or less amounts to just renormalizations.

There are two basic ingredients in obtaining further control however, one geometric and one analytic.

Geometric argument

Limit space $(N, \bar{g}_\infty)$ is Ricci-flat: in scale $\bar{g}_i$,

$$|Ric_{\bar{g}_i}| \leq k \cdot r_h(x_i) \to 0, \quad \text{as} \quad i \to \infty. \tag{2.7}$$

Next, injectivity radius in scale $\bar{g}_i$ satisfies

$$inj_{\bar{g}_i} \geq i_o \cdot r_h(x_i)^{-1} \to \infty, \quad \text{as} \quad i \to \infty. \tag{2.8}$$

So roughly, limit $(N, \bar{g}_\infty)$ has $inj_{g_\infty} = \infty$.

In fact, (2.8) implies $(M, \bar{g}_i)$ contains arbitrarily long, (depending on i), minimizing geodesics, centered at x_i, in all directions. So limit $(N, \bar{g}_\infty)$ has line thru x_∞ in all directions.

Splitting theorem (Cheeger-Gromoll):

A complete manifold (N, g), $\mathrm{Ric}_g \geq 0$, splits isometrically along any line.

Hence,

$$(N, \bar{g}_\infty) = (\mathbb{R}^n, g_o).$$

But, of course,

$$r_h(\mathbb{R}^n, g_o) = \infty.$$

If the convergence $(M, \bar{g}_i, x_i) \to (\mathbb{R}^n, g_o, x_\infty)$ is in **strong** $L^{2,p}$ topology, then the continuity of r_h gives contradiction, since have normalized

$$\bar{r}_h(x_i) = 1$$

.

Idea here: use elliptic regularity to bootstrap or improve smoothness of the convergence.

In harmonic coordinates, Ricci curvature has especially simple form:

$$-\tfrac{1}{2}\Delta g_{\alpha\beta} + Q_{\alpha\beta}(g, \partial g) = Ric_{\alpha\beta}, \tag{2.9}$$

where $\Delta = g^{\alpha\beta}\partial_\alpha\partial_\beta$.

If $r_h(x) = 1$ and $r_h(y) \geq r_o > 0$, $\forall y \in \partial B_x(1)$, then have uniform $L^{1,p}$ bound on Q and uniform $L^{2,p}$ bounds on the coefficients for the Laplacian within $B_x(1 + \frac{1}{2}r_o)$.

Suppose Ric is uniformly bounded in L^∞. Then elliptic regularity applied to (2.9) implies that

$$||g_{\alpha\beta}||_{L^{2,q}} \leq C = C(q),$$

for any $q < \infty$, ($q > p$!).

More importantly, if g_i is a sequence of metrics s.t.

$$(Ric_{g_i})_{\alpha\beta} \Rightarrow (Ric_{g_\infty})_{\alpha\beta},$$

strongly in L^p, then elliptic regularity again implies that

$$(g_i)_{\alpha\beta} \Rightarrow (g_\infty)_{\alpha\beta}.$$

This essentially proves that the $L^{2,p}$ harmonic radius is continuous w.r.t. the strong $L^{2,p}$ topology. Further, when applied to the sequence $\bar{g}_i$ — using $\bar{r}_h \geq 1$ everywhere — it implies that the metrics $\bar{g}_i$ converge strongly in $L^{2,p}$ to the limit $\bar{g}_\infty$.

Lower bound on injectivity radius can be considerably weakened. Define the 1-cross $Cro_1(x)$ of (M, g, x) by

$$Cro_1(x) = sup\{t : \gamma_x(t) = \text{min. geod. on } [-t, t], \gamma(0) = x\}.$$

Then set

$$Cro_1(M, g) = \inf_x Cro_1(x).$$

Has natural analogue in Lorentzian geometry —replace minimizing geodesic by maximizing time-like geodesic.

Theorem 2.2 Convergence II. *Let M be a 4-manifold. Then space of metrics s.t.*

$$|Ric| \leq k, \; Cro_1 \geq c_o, \; vol \geq v_o, \; diam \leq D. \tag{2.10}$$

is precompact in $C^{1,\alpha}$ and weak $L^{2,p}$.

The proof is the same as Theorem 2.1. Lower bound on Cro_1 implies that on blow-up limit $(N, \bar{g}_\infty)$ above, have a line. The splitting theorem then again implies that the limit is flat $\mathbb{R}^4$, giving the same contradiction.

Need volume bound to obtain limit $\mathbb{R}^4$ instead of some quotient $\mathbb{R}^4/\Gamma$ flat manifold.

Of course, in dimension 3 any Ricci-flat manifold is flat. So have $C^{1,\alpha}$ precompactness within the class of metrics on 3-manifolds satisfying

$$|Ric| \leq k, \; vol \geq v_o, \; diam \leq D. \tag{2.11}$$

Remark 2.3 *(i)* The bounds

$$|Ric| \leq k, \quad inj \geq i_o, \quad diam \leq D\,,$$

imply lower bound on r_h.

However, currently there is no proof of an *effective* or *computable* bound. Only proof is by contradiction.

This due to fact there is currently no *quantitative* or *finite* version of the Cheeger-Gromoll splitting theorem —deduce definite geometric bounds on the metric in the presence of (a collection of) minimizing geodesics of a finite but definite length.

If however assume bound on full curvature

$$|Riem| \leq K,$$

then quite easy to obtain computable bound on r_h.

(ii) The proof above can be easily adapted to give a similar result if the L^∞ bound on Ric is replaced by an L^q bound, for some $q > n/2$; one then obtains convergence in weak $L^{2,q}$.

In the opposite direction, the convergence can be improved if one has bounds on the derivatives of the Ricci curvature. This will be the case if Ricci satisfies an elliptic system of PDE, for instance the Einstein equations.

The bounds on injectivity radius in (2.3), or even the 1-cross in (2.10), are rather strong and one would like to replace them with merely a lower volume bound, as in (2.11).

Volume comparison theorem (Bishop-Gromov):

If $Ric \geq (n-1)k$, for some $k > -\infty$, then

$$\frac{volB_x(r)}{volB_k(r)} \downarrow \tag{2.12}$$

$volB_k(r)$ = volume of r-ball in space form of const. curv. k.

In particular, obtain lower bound on volumes of balls on all scales:

$$volB_x(r) \geq \frac{volM}{volB_k(D)} \cdot volB_k(r), \tag{2.13}$$

$D = diamM$.

Cheeger: if bound on Ric strengthened to

$$K_P \geq -K, \ vol \geq v_o, \ diam \leq D, \tag{2.14}$$

where K_P = sectional curvature of any plane P, then

$$inj_g(M) \geq i_o(K, v_o, D).$$

However, this estimate fails under bounds on Ricci.

Example 2.4 Eguchi-Hanson metrics

Let g_λ = family of Eguchi-Hanson metrics on TS^2:

$$g_\lambda = [1 - (\frac{\lambda}{r})^4]^{-1} dr^2 + r^2[1 - (\frac{\lambda}{r})^4]\theta_1^2 + r^2(\theta_2^2 + \theta_3^2), \qquad (2.15)$$

for $r \geq \lambda > 0$.

Locus $r = \lambda$ = image of the 0-section: totally geodesic round $S^2(\lambda)$ of radius λ.

The metrics g_λ are Ricci-flat, and are all homothetic, i.e. are rescalings (via diffeomorphisms) of a fixed metric; in fact,

$$g_\lambda = \lambda^2 \cdot \psi_\lambda^*(g_1), \qquad (2.16)$$

where $\psi_\lambda(r) = \lambda r$. As

$$\lambda \to 0,$$

i.e. as one blows down the metrics,

$$(TS^2, g_\lambda) \to (C(\mathbb{RP}^3), g_0);$$

g_0 = singular flat metric.

Convergence is smooth where $r \geq r_o$, any fixed $r_o > 0$, but is not smooth at $r = 0$. For any $x \in S^2(\lambda)$,

$$inj_{g_\lambda}(x) \to 0.$$

However, volumes of unit balls remains uniformly bounded below.

Metrics g_λ converge to space with orbifold singularity $\mathbb{R}^4/\mathbb{Z}_{\neq}$.

No $C^{1,\alpha}$, or even C^0, compactness.

Large class of Ricci-flat ALE (asymptotically locally Euclidean) spaces, whose metrics are asymptotic to cones $C(S^3/\Gamma)$, $\Gamma \subset SO(4)$, on spherical space forms. This is the family of

ALE gravitational instantons

Gibbons-Hawking, Hawking's Euclidean quantum gravity program.

Theorem 2.5 Convergence III. *Let $\{g_i\}$ be a sequence of metrics on a 4-manifold M, satisfying*

$$|Ric| \leq k,\ vol \geq v_o,\ diam \leq D. \tag{2.17}$$

Then, on a subsequence,

$$(M, g_i) \to (V, g). \tag{2.18}$$

(V, g) = *orbifold, with finite number of singular points* $\{q_j\}$.

• Each q has neighborhood homeomorphic to cone $C(S^3/\Gamma)$, for $\Gamma \subset SO(4)$.
• Metric g is $L^{2,p}$ on regular set

$$V_0 = V \setminus \cup\{q_j\}.$$

•. g extends in a local uniformization of q to C^0 metric on B^4.
• Embeddings $F_i : V_0 \to M$ s.t.

$$F_i^*(g_i) \to g,$$

in weak $L^{2,p}$.
• Convergence in (2.18) is in Gromov-Hausdorff topology, i.e. convergence as metric spaces.

Some important issues in proof:

• Chern-Gauss-Bonnet formula implies

$$\frac{1}{8\pi^2}\int_M |R|^2 dV \leq \chi(M) + C(k, V_o),$$

$C(k, V_o)$ depends only on Ricci curvature bound k and an upper bound V_o on $vol_g M$.

• For each singular point $q \in V$, $\exists$ sequence of rescalings

$$\bar{g}_i = \lambda_i^2 g_i, \quad \lambda_i \to \infty,$$

and base points $x_i \in M$, $x_i \to q$, s.t. on a subsequence

$$(M, \bar{g}_i, x_i) \to (N, \bar{g}_\infty, x_\infty),$$

in $L^{2,p}$, where

$$(N, \bar{g}_\infty) = \text{non-trivial Ricci-flat ALE space.}$$

Any such ALE space has a definite amount of curvature in L^2,

$$\int_N |R|^2 \geq c_0.$$

So only a finite number of such singular points.

Further, the ALE spaces N are embedded in M, in a topologically essential way.

Near singularities, (M, g_i) resembles blow-downs of ALE spaces.

§3. Collapse/Formation of Cusps.

Issue: Behavior of metrics g_i when curvature bounded and

$$vol_{g_i} B_{x_i}(1) \to 0.$$

Throughout this section, assume

$$dim M = 3.$$

Examples. (Product collapse)
$M = S^1 \times V$. Consider curve of metrics

$$g_\lambda = \lambda^2 (d\phi)^2 + g_V.$$

As $\lambda \to 0$, collapse to V.

$Riem_{g_\lambda} = Riem_{g_V}$, so

$$|Riem_{g_\lambda}| \leq K, \text{ as } \lambda \to 0. \tag{3.1}$$

blowing down metric in one direction

(Berger collapse)
On $M = S^3$, consider curve of metrics

$$g_\lambda = \lambda^2 \theta_1^2 + (\theta_2^2 + \theta_3^2).$$

Isometric S^1 action; $L(\text{orbit}) = 2\pi\lambda$.

$$\lambda \to 0 \Rightarrow S^1 \text{ orbit } \to \{pt\}.$$

Main Point:

$$|Riem_{g_\lambda}| \leq K, \text{ as } \lambda \to 0.$$

Collapse to limit space S^2 with bdd. curvature.

Behavior of Taub-NUT metric at Cauchy horizon.

Exactly same procedure works on any n-manifold with

free or locally free S^1.

A Seifert fibered space $N =$ 3-manifold with locally free S^1 action; fibers over surface V with S^1 fibers.

Definition 3.1 A graph manifold G is a closed 3-manifold obtained by glueing Seifert fibered spaces by toral automorphisms of the boundary tori.

Thus,

$$G = S \cup L,$$

$S =$ union of Seifert fibered spaces
$L =$ union of $T^2 \times I$ - glueing region.

On any graph manifold, can construct metrics invariant under S^1/T^2 actions.

Consequence Any compact graph manifold G has volume collapse with bounded curvature:

$$|Ric_{g_i}| \leq k, \quad vol_{g_i}G \to 0. \tag{3.2}$$

If $G = S$, Seifert fibered, can collapse with bounded diameter, $diam_{g_i}S \leq D$. However, if decomposition non-trivial, then

$$diam_{g_i}N \to \infty, \tag{3.3}$$

$N =$ any S or L component. This behavior special to dim 3.

Cheeger-Gromov theory implies converse holds:

Theorem 3.2 *If M is a closed 3-manifold which collapses with bounded curvature, i.e. (3.2) holds, then M is a graph manifold.*

Idea of Proof. First,

$$vol_{g_i}B_x(1) \to 0 \Rightarrow inj_{g_i}(x) \to 0.$$

At any x, rescale inj to size 1,

$$\bar{g}_i = [inj_{g_i}(x)]^{-2} \cdot g_i.$$

Now $|Riem_{\bar{g}_i}| \sim 0$. Thus, metrics $\bar{g}_i$ close to flat metrics on $\mathbb{R}^3/\Gamma \sim \mathbb{R}^2 \times S^1, \mathbb{R} \times S^1 \times S^1$.

Local geometry = geometry on scale of inj, modeled by *non-trivial, flat* 3-manifolds. Glue these local structures consistently.

• Unwrapping collapse.

N = Seifert fibered space, (possibly with boundary). Then

$$\pi_1(S^1) \hookrightarrow \pi_1(N),$$

unless $N = S^3/\Gamma$ or $N = D^2 \times S^1$.

Hence, G a graph manifold, not S^3/Γ and no solid torus in $S \cup L$ decomposition, then

$$\pi_1(orbits) \hookrightarrow \pi_1(N).$$

This means can pass to covering spaces, unwrapping orbits, to unwrap collapse. Obtain convergence to limits in covers.

Further, limits have free isometric S^1 or T^2 action.

Extra symmetry.

Again, this unwrapping special to dim 3.

Formation of cusps.

Mixture of convergence/collapse. Given complete Riemannian manifold (M, g), choose $\varepsilon > 0$ small. Set

$$M^\varepsilon = \{x \in M : volB_x(1) \geq \varepsilon\}, M_\varepsilon = \{x \in M : volB_x(1) \leq \varepsilon\}.$$

$$M^\varepsilon = \varepsilon\text{-thick part, } M_\varepsilon = \varepsilon\text{-thin part.}$$

Now let $\{g_i\}$ = sequence of complete metrics on M.

- If $x_i \in M^\varepsilon$, then have convergence in regions about x_i.
- If $y_i \in M_{\varepsilon_o}$, then region about y_i is graph manifold.
- If $z_i \in M_{\varepsilon_i}$, $\varepsilon_i \to 0$, then region about z_i collapsing.

Theorem 3.3 Cusp Formation *M a closed 3-manifold and g_i sequence of metrics on M satisfying*

$$M^\varepsilon \neq \emptyset, \quad M_\varepsilon \neq \emptyset, \quad \forall\varepsilon.$$

Then pointed subsequences (M, g_i, p_i) converge to:

- *complete cusps N - open 3-manifolds with graph manifold ends.*
- *collapsed graph manifolds of infinite diameter.*

Remark 3.4

(i). There are versions of these convergence/collapse/cusp results also in dimension 4, (as well as in higher dimensions).

The concept of graph manifold is generalized to manifolds having an "F-structure". Here, as in the orbifold degeneration result, one must allow for a finite number of point singularities in F-structure.

(ii). There are also versions of these results with L^p bounds on the Ricci curvature in place of L^∞, provided $p > n/2$, $n = dim M$.

§4. Applications to Static and Stationary Space-Times.

Apply convergence/collapse theory to (vacuum) stationary space-times $(\mathbf{M},\ \mathbf{g})$.

Time-independent - so can use directly methods of Riemannian geometry.

K = time-like Killing field. Assume throughout

$$(\mathbf{M},\ \mathbf{g}) \text{ chronological, } K \text{ complete.}$$

Σ = orbit space of $\mathbb{R}$-action, $\pi : \mathbf{M} \to \Sigma$ principal $\mathbb{R}$ bundle.
4-metric $\mathbf{g}$ of form:

$$\mathbf{g} = -u^2(dt + \theta)^2 + \pi^*(g); \tag{4.1}$$

$K = \partial/\partial t$, θ connection 1-form for bundle π, $u^2 = -\mathbf{g}(K, K) > 0$ and $g = g_\Sigma$ = metric induced on orbit space.

Define twist 1-form ω by

$$2\omega = *(\kappa \wedge d\kappa) = -u^4 * d\theta,$$

$\kappa = -u^2(dt + \theta)$, 1-form dual to K.

The vacuum Einstein equations

$$Ric_{\mathbf{g}} = 0,$$

equivalent to elliptic system of P.D.E's in the data (g, u, ω):

$$Ric_g = u^{-1}D^2u + 2u^{-4}(\omega \otimes \omega - |\omega|^2 g), \tag{4.2}$$

$$\Delta u = -2u^{-3}|\omega|^2, \tag{4.3}$$

$$d\omega = 0. \tag{4.4}$$

Σ open, possibly with boundary. Locally, to obtain uniqueness, need to impose boundary conditions.

• Global issues.

(Σ, g) = complete, non-compact 3-manifold.

Boundary conditions then at infinity.

Theorem 4.1 (Lichnerowicz). *The only complete, stationary vacuum space-time* $(\mathbf{M}, \mathbf{g})$ *which is asymptotically flat (AF) is empty Minkowski space-time* $(\mathbf{R}^4, \eta)$.

Stationary space-times model isolated physical systems. Only physically realistic models are AF.

• Physically, Lichnerowicz theorem a triviality. No source for the gravitational field, it must be empty.

• Mathematically, not (so) trivial. In fact

AF assumption ad-hoc, counter to spririt of GR

Theorem 4.2 (Generalized Lichnerowicz). *The only complete stationary vacuum space-time* $(\mathbf{M}, \mathbf{g})$ *is empty Minkowski space-time* $(\mathbf{R}^4, \eta)$*, or a discrete isometric quotient of it.*

Outline of Proof:

• Study moduli space of complete stationary vacuum solutions, any asymptotic behavior.

Curvature could be unbounded at infinity. If so, can find base points and rescalings to obtain new stationary vacuum solution, (i.e. a new point in the moduli space), with

$$|Ric_{\Sigma}|(x) = 1, \ |Ric_{\Sigma}|(y) \leq K, \forall y.$$

• Ernst formulation. Define the conformally related metric $\widetilde{g}$:

$$\widetilde{g} = u^2 g. \tag{4.5}$$

Equation for Ric then becomes:

$$Ric_{\widetilde{g}} = 2(d \ln u)^2 + 2u^{-4}\omega^2 \geq 0. \tag{4.6}$$

Elliptic system (4.2)-(4.4) becomes the Euler-Lagrange equations for

$$S_{\text{eff}} = \int [R - \frac{1}{2}(\frac{|d\phi|^2 + |du^2|^2}{u^4})],$$

ϕ = twist potential, $d\phi = 2\omega$.

3-d gravity coupled to σ-model, target = $(H^2(-1), g_{-1})$.

Thus, Ernst map

$$E = (\phi, u^2) \tag{4.7}$$

is harmonic map

$$E : (\Sigma, \widetilde{g}) \to (H^2(-1), g_{-1}).$$

Harmonic maps $E : (M, g) \to (N, h)$ with domain $Ric \geq 0$, target of $curv \leq 0$ have strong rigidity properties, via the Bochner-Lichnerowicz formula,

$$\frac{1}{2}\Delta|DE|^2 = |D^2E|^2 + < Ric_g, E^*(h) > - \sum(E^*R_h)(e_i, e_j, e_j, e_i). \tag{4.8}$$

Analyse this carefully:

show E is a constant map, and so $(\mathbf{M},\ \mathbf{g})$ is flat.

Remark 4.3
(i). Same result and proof holds for stationary gravitational fields coupled to σ-models, with target spaces = Riemannian manifolds of non-positive sectional curvature.

(ii). Open Problem.

Riemannian analogue of generalized Lichnerowicz.

Thus, does there exist a non-flat Ricci-flat Riemannian 4-manifold which admits a *free* isometric S^1 action?

• Local issues.

This rigidity $\Rightarrow$ apriori estimates on geometry of general stationary (vacuum) solutions.

Suppose Σ not complete, so $\partial\Sigma \neq \emptyset$.

Part of $\partial\Sigma$ may correspond to horizon $H = \{u = 0\}$.

Theorem 4.4 (Curvature Estimate). *Let $(\mathbf{M}, \mathbf{g})$ be a stationary vacuum space-time. Then there is a constant $C < \infty$, independent of $(\mathbf{M}, \mathbf{g})$, such that*

$$|\mathbf{R}|(x) \leq C/r^2[x], \tag{4.9}$$

where $r[x] = dist_{\Sigma}(\pi(x), \partial\Sigma)$.

Here, the curvature norm $|\mathbf{R}|$ may be given by

$$|\mathbf{R}| = |R_{\Sigma}| + |d \ln u|^2 + |u^{-2}\omega|^2.$$

Remark 4.5 (i). Using elliptic regularity, one also has higher order bounds:

$$|\nabla^k \mathbf{R}|(x) \leq C_k/r^{2+k}[x]. \tag{4.10}$$

(ii). A version of this result also holds for stationary space-times with energy-momentum tensor T. Thus, for example one has

$$|\mathbf{R}|(x) \leq C_{\alpha} \cdot |T|_{C^{\alpha}(B_{[x]}(1))}, \tag{4.11}$$

for any $\alpha > 0$, where $B_{[x]}(1)$ is the unit ball in (Σ, g) about $[x]$.

Thus, one can use the Cheeger-Gromov theory to control local behavior of stationary space-times, away from any boundary.

• Asymptotic behavior.

Study apriori possible asymptotic behavior of stationary vacuum solution. Know for instance,

curvature decays as r^{-2} at ∞.

Restrict to static space-times $(\mathbf{M}, \mathbf{g})$

Orbit space (Σ, g). Define $\partial\Sigma$ to be *pseudo-compact* if $\exists r_o > 0$ s.t. level set $\{r = r_o\}$ in Σ is compact.

Let $S(s) = r^{-1}(s) \subset \Sigma$. If E is an end of (Σ, g), set

$$m_E = lim_{s \to \infty} \tfrac{1}{4\pi} \int_{S(s)} < \nabla \ln u, \nabla t > dA. \tag{4.12}$$

Theorem 4.6 (Static Asymptotics). *$(\mathbf{M}, \mathbf{g})$ a static vacuum space-time with pseudo-compact boundary. Then*
- *$(\mathbf{M}, \mathbf{g})$ has a finite number of ends.*
- *Any end E on which $\liminf_E u > 0$, is either:*

AF

or

$$\text{small} \equiv \int_1^\infty areaS(r)^{-1} dr = \infty. \tag{4.13}$$

- *If $m_E \neq 0$ and $sup_E u < \infty$, then E is AF.*

When E is AF, it is AF in usual "strong" sense:

$$|g - g_0| = \frac{2m}{r} + O(r^{-2}),\ |R| = O(r^{-3}),\quad |u - 1| = \frac{m}{r} + O(r^{-2}).$$

Ideas of Proof:

Study asymptotic behavior of an end E by "blowing it down".

R large, k fixed: consider annuli $A(R, kR)$ about $x_o \in (\Sigma, g)$. In rescalings

$$g_R = R^{-2} g,$$

$A(R, kR)$ becomes metric annulus $A(1, k)$ w.r.t. g_R. Quadratic curvature decay $\Rightarrow$ curvature of g_R uniformly bounded.

Thus, apply the Cheeger-Gromov theory to a sequence $(A(1, k), g_{R_i})$, $R_i \to \infty$.

Convergence (or cusp) case gives AF ends, collapse case gives small ends.

For small ends, obtain an extra S^1 or T^2 symmetry when collapse is unwrapped in covering spaces. Asymptotic behavior then described by axisymmetric static solutions, i.e. the Weyl metrics.

Remark 4.7 There exist static vacuum solutions, *smooth* up to the horizon, which have a single *small* end.

Myers metrics = periodic Schwarzschild metrics.

$\Sigma = (D^2 \times S^1) \setminus S^2$, $\partial\Sigma = S^2$, end $T^2 \times \mathbb{R}^+$, asymptotic to a (static) Kasner metric.

This is of course not a counterexample to the static black hole uniqueness theorem, since the end is not AF.

§5. Lorentzian Analogues and Open Problems

Issue: Apply convergence/collapse theory to time-independent space-times $(\mathbf{M}, \mathbf{g})$.
Main Focus: Vacuum space-times

$$Ric_{\mathbf{g}} = 0,$$

or at least $|Ric_{\mathbf{g}}| \leq K$.

• Motivation Global stability results:

1. Minkowski space-time (Christodoulou-Klainerman)
2. de Sitter space-time (Friedrich)
3. Milne space-time (Andersson-Moncrief)
4. $U(1)$ Bianchi model (Choquet-Bruhat & Moncrief)

openness results

Basic features of given model preserved under (suitable) small perturbations of initial data.
What happens to limits of such perturbations?

• Difficulties.
(i). Ricci curvature:

Elliptic PDE for Riem. metrics $\rightarrow$ Hyperbolic PDE for Lor. metrics

(ii).

$$O(4) \text{ compact} \rightarrow O(3,1) \text{ non-compact}$$

1^{st} Level Problem

Bound local geometry in terms of

$$|\mathbf{R}|_{L^\infty} \leq K. \tag{5.1}$$

Usual norm of curvature tensor

$$|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \text{ not } \geq 0. \tag{5.2}$$

For Ricci-flat Lorentz metrics, 2 scalar invariants of full curvature

$$|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \text{ and } <\mathbf{R}, *\mathbf{R}>= \mathbf{R}_{ijkl}(*\mathbf{R}^{ijkl}).$$

Both can vanish on non-flat space-times: e.g.

plane-fronted gravitational waves

$$\mathbf{g} = -2dudv + 2(dx^2 + dy^2) - 2h(u,x,y)du^2,$$

$$\Delta_{(x,y)}h = 0, \; h \text{ arbitrary in } u.$$

Class of such highly non-compact $\Rightarrow$ no local control of metric in any coord. system, under bounds (5.2).

Thus, must consider bounds on $\mathbf{R}$ w.r.t. a framing or coordinate system.

Let $T = e_0 =$ unit time-like vector, future directed. Extend to o.n. frame e_α, $0 \leq \alpha \leq 3$. $T^\perp$ space-like, $O(3)$ compact, so framing of $T^\perp$ unimportant. Define

$$|\mathbf{R}|_T^2 = \sum (\mathbf{R}_{ijkl})^2; \tag{5.3}$$

components w.r.t. frame e_α. Equivalent to taking the norm of $\mathbf{R}$ w.r.t. the Riemannian metric

$$g_E = \mathbf{g} + 2T \otimes T.$$

As long as T stays in a compact subset of $T^+\mathbf{M} =$ bundle of future interior null cones, norms (5.3) all equivalent.

Noted earlier:
$(M, g) =$ smooth Riemannian manifold with L^∞ bound on the full curvature,

$$|R| \leq K$$

then $\exists$ charts in which have $L^{2,p}$ control of metric; bounds on $g_{\alpha\beta}$ and radius of charts depend only on K and lower volume bound.

Size conditions. Let Ω = domain in a smooth Lorentz manifold $(\mathbf{M}, \mathbf{g})$, with smooth time function $T = \partial/\partial t$. Let $S = t^{-1}(0)$ and suppose the 1-cylinder

$$C_1 = B_p(1) \times [-1, 1] \subset\subset \Omega.$$

Let $D = ImT|_{C_1} \subset\subset T^+\Omega$.

Theorem 5.1 *Suppose Ω satisfies the size conditions and $\exists$ constants $K < \infty$, $v_o > 0$ s.t.*

$$|\mathbf{R}|_T \leq K, \quad vol_g B_p(1) \geq v_o. \tag{5.4}$$

Then $\exists r_o > 0$, $R_o < \infty$, depending only on K, v_o and D, and coord. charts on the r_o-cylinder

$$C_{r_o} = B_p(r_o) \times [r_o, r_o] \subset C_1,$$

s.t. on C_{r_o},

$$||\mathbf{g}_{\alpha\beta}||_{L^{2,p}} \leq R_o. \tag{5.5}$$

Result formulated so easy to pass to limits:

Given sequence of smooth space-times $(\mathbf{M}_i, \mathbf{g}_i)$ satisfying hypotheses of Theorem.

Then, in a subsequence, $\exists$ limit $C^{1,\alpha} \cap L^{2,p}$ space-time $(\mathbf{M}_\infty, \mathbf{g}_\infty)$, defined at least on the cylinder C_{r_o}.

Further, the convergence to the limit is $C^{1,\alpha}$ and weak $L^{2,p}$.

Result holds in dim $= n + 1$.

Idea of Proof: Construct a new time function τ on a small cylinder C_{r_o} with $|\nabla\tau|^2 = -1$, so the flow of $\nabla\tau$ is by time-like geodesics. On the level sets S_τ of τ, construct local harmonic coordinates x_1, x_2, x_3, (w.r.t. induced Riemannian metric).

Gives local coordinate system (τ, x_1, x_2, x_3) on C_{r_o}. Use Raychaudhuri equation, Bochner-Weitzenbock formula, (Simons' equation), and elliptic estimates to control $\mathbf{g}_{\alpha\beta}$.

If lower volume bound on $B_p(1) \subset S$ is dropped, then S may collapse with bounded curvature. Examples of this behavior occur on approach to Cauchy horizons, (Taub-NUT example).

Rendall: if Σ is a *compact* Cauchy surface in say smooth vacuum space-time, then nearby space-like hypersurfaces collapse with bounded curvature.

2^{nd} Level Problem

Replace $|\mathbf{R}|_T$ bound by bound on $|Ric_{\mathbf{g}}|$.

For vacuum space-times: remove bound on $|\mathbf{R}|_T$.

Seek analogues of Convergence I, II results:

bound on Ric, inj/1-cross $\Rightarrow L^{2,p}$ control on g

Two parts to proof: geometric/analytic

geometric

• Splitting theorem
Have direct analogue.

Lorentzian Splitting Theorem (Eschenburg, Galloway, Newman) $(\mathbf{M},\ \mathbf{g})$ time-like geodesically complete or globally hyperbolic vacuum space-time which contains a time-like line, then $(\mathbf{M},\ \mathbf{g})$ is flat.

Can define Lorentzian 1-cross: $|T|^2 = -1$

$$Cro_1(x,T) = \sup\{t : \gamma = \text{max. geod. on}[-t,t], \gamma(0) = x, \gamma'(0) = T\},$$

$$Cro_1(\Omega,T) = \inf_{x\in\Omega} Cro_1(x,T).$$

Missing step - No regularity boost from hyperbolic PDE.

However, smoothness of initial data preserved until hit boundary of maximal development.

Let $S \subset (\mathbf{M}, \mathbf{g})$ = space-like hypersurface. Define H^s harmonic radius $\rho_s(x)$, $x \in S$, $s > 2.5$, (large) as before: largest radius s.t.

$$[\rho_s(x)]^{2s-3} \int_{B_x(\rho_s(x))} |\partial^s g_{\alpha\beta}|^2 \leq C.$$

Suppose S = hypersurface with smooth, (C^∞), initial data. Let S_t = hypersurface obtained from vacuum evolution. Then (Choquet-Bruhat)

$$\min_{x_t \in S_t} \rho_s(x_t) \geq c_1 \Rightarrow \min_{x_t \in S_t} \rho_{s+1}(x_t) \geq c_2, \tag{5.6}$$

where c_2 depends on c_1 and the initial data set.

We raise the following:

Regularity Problem. Can the estimate (5.6) be improved to an estimate

$$\min_{x_t \in S_t} \rho_{s+1}(x_t) \geq c_0 \min_{x_t \in S_t} \rho_s(x_t), \tag{5.7}$$

where c_0 depends only on the initial data set?

The important point of (5.7) over (5.6) is that the estimate (5.7) is scale-invariant.

If (5.7) holds, it serves as an analogue of the regularity boost. Can then imitate the proofs of Riemannian convergence results to obtain Lorentzian convergence.

Would have numerous interesting applications.

Next, drop any assumption on the 1-cross of $(\mathbf{M},\ \mathbf{g})$: maintain only a lower bound on the volumes of geodesic balls on space-like hypersurfaces.

This leads to issues of singularity formation and the structure of the boundary of the vacuum space-time; little understood mathematically.

Sandwich Problem: Version I.

Let $(\mathbf{M}, \mathbf{g}_i)$ be a sequence of vacuum space-times, and let Σ_i^1, Σ_i^2 be two compact Cauchy surfaces in $\mathbf{M}$, with $\Sigma_i^2 \gg \Sigma_i^1$ and with

$$1 \leq dist_{\mathbf{g}}(x, \Sigma_i^1) \leq 10, \quad \forall x \in \Sigma_i^2.$$

Suppose the Cauchy data (g_i^j, K_i^j), $j = 1, 2$ on each Cauchy surface are uniformly bounded in H^s, for some fixed $s > 2.5$, possibly large. Hence the data (g_i^j, K_i^j) converge, in a subsequence and weakly in H^s, to limit H^s Cauchy data g_∞^j, K_∞^j on Σ^j.

Do the vacuum space-times $A_i(1,2) \subset (M, g_i)$ between Σ^1 and Σ^2 converge, weakly in H^s, to a limit space time,

$$(A_i(1,2), g_i) \to (A_\infty, g_\infty)? \tag{5.8}$$

Application. Understand limits of the AS vacuum perturbations of deSitter, (Friedrich).

The sandwich problem asks: suppose one has control on the space-time near past **and** future space-like infinity $\mathcal{I}^{\pm}$, does it follow that one has control in between?

Sandwich Problem: Version II. ($\mathbf{M}$, $\mathbf{g}$) smooth vacuum space-time, Σ^1, Σ^2 smooth compact space-like hypersurfaces,

$$\Sigma^1 \gg \Sigma^2.$$

Does the maximal globally hyperbolic development from Σ^1 contain Σ^2?

Can ($\mathbf{M}$, $\mathbf{g}$) have an "invisible" singularity in between?

§6. Future Asymptotics and Geometrization of 3-Manifolds

Issue: Understand future asymptotics of vacuum cosmological space-times $(\mathbf{M},\ \mathbf{g})$.

$(\mathbf{M},\ \mathbf{g})$ contains compact CMC Cauchy surface Σ,

$$\sigma(\Sigma) \leq 0. \tag{6.1}$$

$\exists$ foliation $\mathcal{F}$ by CMC Cauchy surfaces $\Sigma_\tau \approx \Sigma$,

$$\tau = \text{mean curvature } \in (-\infty, 0). \tag{6.2}$$

$\tau \uparrow$ to future – expanding direction. Let $\mathbf{M}_{\mathcal{F}}$ = foliated region in $\mathbf{M}$.

Suppose $(\mathbf{M},\ \mathbf{g})$ geodesically complete to future of Σ and, to future of Σ,

$$\mathbf{M} = \mathbf{M}_{\mathcal{F}}.$$

Strong assumptions, but necessary to study smooth asymptotics.

Induced metrics $g_\tau = \mathbf{g}|_\Sigma$ = curve of Riemannian metrics on fixed 3-manifold Σ. As $\tau \to 0$, typically

$$vol_{g_\tau}\Sigma \to \infty,\ \ g_\tau \text{ becomes flat}$$

– due to expansion.

Not so interesting. To study asymptotics, rescale by distance tp fixed base point – "blow-down".

For $x \gg \Sigma$, let

$$t(x) = dist_{\mathbf{g}}(x, \Sigma)$$

and

$$t_\tau = t_{max}(\tau) = max\{t(x) : x \in \Sigma_\tau\}. \tag{6.3}$$

Thus study the asymptotic behavior of the metrics

$$\bar{g}_\tau = t_\tau^{-2} g_\tau, \tag{6.4}$$

on Σ_τ. For rescaled space-time $(\mathbf{M}, \bar{\mathbf{g}}_\tau)$, distance of $(\Sigma_\tau, \bar{g}_\tau)$ to "big bang" $\to 1$, as $\tau \to 0$.

Definition 6.1 Σ a closed, oriented, connected 3-manifold, non-positive Yamabe type. A *weak* geometrization of Σ is a decomposition

$$\Sigma = H \cup G : \tag{6.5}$$

• H = finite collection of complete, connected hyperbolic manifolds, of finite volume $\subset \Sigma$.
• G = finite collection of connected graph manifolds $\subset \Sigma$.
• Union along a finite collection of tori $\mathcal{T} = \cup T_i = \partial H = \partial G \subset \Sigma$.

A *strong* geometrization of Σ is a weak geometrization s.t.

$$\pi_1(T^2) \hookrightarrow \pi_1(\Sigma), \ \forall T^2 \in \mathcal{T}.$$

$\mathcal{T} = \emptyset \Rightarrow$ weak = strong.

$\exists$ sequences of metrics g_i which limit on a geometrization of Σ.
• $g_i \to$ hyperbolic metric on H
• $g_i \to$ collapse on G
match behaviors far down hyperbolic cusps.

Curvature assumption. Assume $\exists C < \infty$ s.t. for $x \gg \Sigma$,

$$|\mathbf{R}|(x) + t(x)|\nabla \mathbf{R}|(x) \leq C \cdot t^{-2}(x). \tag{6.6}$$

Curvature norm $|\mathbf{R}| = |\mathbf{R}|_T$, T = unit normal to the foliation Σ_τ. Bound (6.6) scale-invariant.

- holds for Bianchi space-times
- Conjecture: holds for perturbations of Bianchi space-times
- probably holds for Gowdy

No known cosmological vacuum space-times, geodesically complete to future, where it fails.

Theorem 6.2 *Let* $(\mathbf{M}, \mathbf{g})$ *be a cosmological space-time of non-positive Yamabe type. Suppose that the curvature assumption (6.6) holds, and that* $M_{\mathcal{F}} = \mathbf{M}$.

Then $(\mathbf{M}, \mathbf{g})$ *is future geodesically complete and, for any sequence* $\tau_i \to 0$, *the slices* $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$ *have a subsequence converging to a weak geometrization of* Σ.

Ideas in Proof: Curve of metrics $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$, blown-down.

- L^∞ bound on $|\mathbf{R}|_T \Rightarrow L^\infty$ bound on intrinsic and extrinsic curvature of slices $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$. Proof similar to Theorem 5.1.
- Can apply Cheeger-Gromov theory: subsequences either converge, collapse or form cusps.

Collapse/cusps $\Rightarrow$ graph manifold structure.

Remains to show that convergence/thick part converges always to hyperbolic manifolds.

Main ingredient for convergence: Volume monotonicity:

$$\frac{vol_{g_\tau}\Sigma_\tau}{t_\tau^3}\downarrow, \tag{6.7}$$

Analogous to monotonicity of reduced Hamiltonian (Fischer-Moncrief).

This monotonicity follows from simple analysis of the Raychaudhuri equation, (as in Penrose-Hawking singularity theorems), together with a suitable maximum principle.

Further, the ratio in (6.7) is constant on some interval $[\tau_1, \tau_2]$ iff the annular region $A(\tau_1, \tau_2) = \tau^{-1}(\tau_1, \tau_2) =$ annulus in a *flat* Lorentzian cone

$$\mathbf{g_o} = -dt^2 + t^2 g_{-1},$$

where g_{-1} is a hyperbolic metric.

Again, the ratio in (6.7) is scale invariant, and so

$$\frac{vol_{g_\tau}\Sigma_\tau}{t_\tau^3} = vol_{\bar{g}_\tau}\Sigma_\tau.$$

Non-collapse means this volume is bounded below. Hence, it converges to a non-zero limit, so have equality in limit, so space-time limit is flat Lorentz cone.