Future complete U(1) symmetric Einsteinian spacetimes, the unpolarized case.

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1 Introduction.

In this paper I generalize the non linear stability theorem obtained in collaboration with V. Moncrief (CB-M1, CB-M2) for vacuum Einsteinian 4 - manifolds $(V,^{(4)}g)$ where $V = M \times R$ with M a circle bundle over a compact, orientable surface Σ of genus greater than 1. The Lorentzian metric $^{(4)}g$ admits a Killing symmetry along the (spacelike) circular fibers. I remove the so-called polarization condition, i.e. the orthogonality of the fibers to quotient 3-manifolds. The reduced field equations take now the form of a wave map equation, instead of a linear wave equation in the polarized case, coupled to 2+1 gravity. I use results on wave maps from curved manifolds obtained in CB1, CB2. Like in CB-M2 we do not restrict the conformal geometry of Σ to avoid those regions of Teichmüller space for which the lowest positive eigenvalues of the scalar Laplacian lie, in our normalization, in the gap $(0, \frac{1}{8}]$. A consequence is that the asymptotic behaviour of the wave map field does not exhibit a universal rate of decay but instead develops a decay rate which depends upon the asymptotic values of the lowest eigenvalue.

Under the Kaluza-Klein reduction which one carries out in the presence of the assumed spacelike Killing field one is first led to field equations of the type of an Einstein - Maxwell - Jordan system on the 3 - manifold $\Sigma \times R$. To transform this to a more convenient Einstein - wave map system one needs a further topological restriction on the fields allowed. The need for this arises from considering the constraint equation for the effective 2+1 dimensional electric type field density $\tilde{e} = e^a \frac{\partial}{\partial x^a}$ which reads $e^a_{,a} = 0$. On a higher genus surface Σ the general solution of this equation (which results from a Hodge decomposition of one - forms on Σ) takes the form $e^a = \varepsilon^{ab}(\omega,_b + h_b)$ where $h_b dx^b$ is a harmonic one - form on Σ. A consistent simplification results from setting this harmonic contribution to zero so that $e^a \frac{\partial}{\partial x^a}$ becomes expressible purely in terms of the so - called twist potential ω. Taken together with the norm of the ($U(1)$ generating) Killing field $\tilde{Y}$, conveniently expessed via $\tilde{Y}.\tilde{Y} = e^{2\gamma}$, the twist potential ω and the function γ provide a map from $\Sigma \times R$ to R^2. When expressed in terms of the pair (γ, ω) Einstein's equations take the form of a wave map from a 2+1 Lorentzian manifold $(\Sigma \times R, {}^{(3)} g)$ into the Poincaré plane with its standard metric $2d\gamma^2 + \frac{1}{2}e^{-4\gamma}d\omega^2$; the metric ${}^{(3)}g$ satisfies the 2+1 Einstein equations on $\Sigma \times R$, with source the wave map. These 2+1 Einstein equations, supplemented by suitable coordinate conditions to fix the gauge, reduce to an elliptic system on each slice Σ_t of $\Sigma \times R$ for the lapse, the shift, and the conformal factor of a 2 dimensional metric, together with an ordinary differential system for the Teichmüller parameters with determine the conformal geometry. The wave maps fields and the Teichmüller parameters represent therefore the true propagating gravitational degrees of freedom of the original problem.

The basic methods we use to prove existence for an infinite proper time involve the construction of higher order energies to control the Sobolev norms of the wave map combined with an application of the "Dirichlet energy" function in Teichmüller space to control the Teichmüller parameters degrees of freedom. A subtlety is that the most obvious definition of wave map energies does not lead to a well defined rate of decay so that suitable corrected energies must be developed which exploit information about the lowest eigenvalues of the spatial Laplacian which appears in the relevant wave operators. The eigenvalues vary with position in Teichmüller space and thus evolve along with Teichmuller parameters. If the lowest (non trivial) eigenvalue asymptotically avoids a well known gap in the spectrum (the gap $(0, \frac{1}{8}]$ in our normalization which has the more familiar form $(0, \frac{1}{4}]$ if one instead normalizes the Gauss curvature on the higher genus surface) then we obtain a universal rate of decay for the energies asymptotically. If the lowest eigenvalue however drifts into this gap and remains there asymptotically then the rate of decay of the energies will depend upon the asymptotic value of this lowest eigenvalue and will no longer be universal. We need slightly different forms for the corrected energies to handle these different eventualities (universal versus non universal rates of decay). In all cases the conformal geometry of our circle bundles undergoes a kind of Cheeger - Gromov collapse in which the circular fibers (after a conformal rescaling needed to take out the overall expansion) collapse to zero length asymptotically while only the conformal 2 - geometry remains well behaved. In our set up the Sobolev constants depend only on the conformal 2 - geometry (i.e. upon the Teichmüller parameters) and, so long as the evolution remains in a compact subspace of Teichmüller space, these constants remain under control.

The sense in which our solutions are global in the expanding direction is that they exhaust the maximal range allowed for the mean curvature function on a manifold of negative Yamabe type, for which a zero mean curvature cannot be achieved but only asymptotically approached. In addition however our estimates prove that the normal trajectories to our spatial slices all have infinite future proper time length, and allow us to establish, using [CB-C], causal geodesic completeness in the expanding direction.

If the harmonic one - form discussed above were allowed to be non zero it would disturb the pure wave map character of the reduced field equations. On the other hand it seems plausible that energy arguments could still be made to work in the presence of h. Alternatively one might simply refrain from trying to force the reduced field equations into a wave map framework and instead develop energy arguments for the Einstein - Maxwell - Jordan type system itself which require no splitting of $\tilde{e}$ into twist potential and harmonic contributions. We shall not however pursue either of these possibilities here but leave them for further study.

I need as in CB-M1 and CB-M2 a smallness condition on suitably defined energies which control the norms of the evolving (herere wave map) field and for this reason I continue to restrict my attention to trivial S^1 bundles over Σ (i.e., to those for which $M = S^1 \times \Sigma$). The reason for this is that the curvature of the $U(1)$ connection and its assumed ($U(1)$ - generating) Killing field has a quantized integral over Σ and, in the case of a non trivial bundle when this integral is not zero, cannot be adjusted to satisfy the smallness condition needed for the energy argument. It seems plausible that one could probably substract off this unavoidable topological contribution to curvature and work with suitable energies defined for the substracted fields to handle the case of non trivial bundles but I shall not attempt to do so here.

Another approach (suggested by V. Moncrief) to treating solutions on non trivial S^1 bundles involves applying a well known action of $SL(2, R)$ (the isometry goup of the Poincaré plane which plays the role of target for our wave map fields) to the fields defined on the base manifold $\Sigma \times R$. In certain cases this group action can be used to transform solutions which lift to the trivial S^1 bundle over $\Sigma \times R$ to other solutions which lift instead to another, non trivial bundle. There is an obstruction to obtaining such solutions in this way since a certain Casimir invariant (which is of course preserved under the group action) is necessarily positive for solutions which lift to the trivial bundle (it can be negative for a subset of solutions which lift to non trivial bundles). This formulation has so far only been developed for the case of circle bundles over $S^2 \times R$ but can most likely be generalized to the cases of bundles over $\Sigma \times R$ where Σ is either a torus or a higher genus surface. That possibility is left for further study.

The small data future global existence theorem for solutions of Einstein's equations of Andersson and Moncrief, this volume, makes no symmetry assumption whatsoever, but treats a different class of spatial 3 - manifolds which are taken to be compact hyperbolic. The results of their analysis show that the standard hyperbolic (i.e. constant negative curvature) metric on such a manifold serves as an attractor for the conformal geometry under the (future) Einstein flow. In other words the evolving conformal geometry has a well behaved limit in that problem. This fact plays a crucial role in their analysis since various Sobolev "constants" (which are in fact functionals of the geometry) which are needed in the associated energy estimates are asymptotically under control since they are tending toward their (regular) limiting values for the hyperbolic metric. Thus the difficulty of degenerating Sobolev constants, avoided by the introduction of a conformal 2 metric in the case of our $U(1)$ symmetry assumption, never arises in the Andersson - Moncrief work.

Besides the fact that the $U(1)$ symmetric case is not included in the no symmetry case treated by Andersson and Moncrief, an interest of the $U(1)$ case is that in our problem the number of effective spatial dimensions is two, and also that there is no known "physical" reason why large data solutions should develop singularities in the direction of cosmological expansion. Black hole formation seems to be suppressed by the topological character of the assumed Killing symmetry (which is of translational rather than rotational type and excludes the appearance of an axis of symmetry) and the big bang singularity is avoided by considering the future evolution from an initially expanding Cauchy hypersurface. Any possible big crunch is excluded by our requirement that the spatial manifold M is of negative Yamabe type (which is true of all circle bundles over higher genus manifolds). Such manifolds are incompatible (in the vacuum and electrovacuum cases for example) with the development of a maximal hypersurface which would be a necessary prelude to the "recollapse" of an expanding universe towards a hypothetical big crunch singularity. At a maximal hypersurface the scalar curvature of M would have to be everywhere positive - an impossibility on any manifold of negative Yamabe type. Thus it is conceivable that for large data future global existence holds for our problem. Up to now the only large data global results require simplifying assumptions so stringent that they effectively reduce the number of spatial dimensions to one (e.g., Gowdy models and their generalizations, plane symmetric gravitational vaves, spherically symmetric matter coupled with gravity) or zero (e.g. Bianchi models, 2+1 gravity). Unfortunately we have at present no way of proving this global existence, even in the polarized case for which the wave map equation reduces to a wave equation for a scalar function, because the reduced field equations are non local in character. The "background" spacetime on which the scalar field evolves is not given a priori but is instead a certain functional (obtained by solution of elliptic equations) of the evolving field (and the Teichmüller parameters) itself. In the unpolarized case, the problem of global existence of strong solutions for wave maps on a fixed background in 2+1 dimensions is still unsolved. However there is a proof [M-S] of global existence of a weak solution (with no uniqueness) for wave maps from Minkowski spacetime. Any progress on the large data global existence, even of weak solutions, for the $U(1)$ - symmetric problem would represent a "quantum jump" forward in our understanding of long time existence problems for Einstein's equations.

It is worth mentioning here that, again with suitable topological restrictions, using the reduction obtained by V. Moncrief (M2), an analogous Einstein - wave map form of the reduced field equations can be obtained even when one begins with the full Einstein - Maxwell system in 3+1 dimensions.

Some steps of the proof given here have been obtained independently, using other notations, by V. Moncrief. I thank him for communicating his manuscript to me, and for numerous conversations on the subject.

2 S^1 invariant einsteinian universes.

2.1 Definition.

The spacetime manifold V is a principal fiber bundle with Lie group S^1 and base $\Sigma \times R$, with Σ a smooth orientable 2 dimensional manifold which we suppose here to be compact and of genus greater than one.

The spacetime metric ${}^{(4)}g$ is invariant under the action of S^1, the orbits are the fibers of V and are supposed to be space like. We write it in the form

$$ {}^{(4)}g = e^{-2\gamma(3)}g + e^{2\gamma}(\theta)^2, $$

where γ is a scalar function and ${}^{(3)}g$ a lorentzian metric on $\Sigma \times R$ which reads:

$$ {}^{(3)}g = -N^2 dt^2 + g_{ab}(dx^a + \nu^a dt)(dx^b + \nu^b dt) $$

equivalently, in terms of a moving frame

$$ {}^{(3)}g = -N^2(\theta^0)^2 + g_{ab}\theta^a\theta^b, \quad \theta^0 \equiv dt, \quad \theta^a \equiv dx^a + \nu^a dt $$

N and ν are respectively the lapse and shift of ${}^{(3)}g$,while

$$ g = g_{ab} dx^a dx^b $$

is a riemannian metric on Σ, depending on t.

The 1-form θ is a connection on the fiber bundle V, represented in coordinates (x^3, x^α) adapted to a local trivialization of the bundle by

$$ \theta = dx^3 + A_\alpha dx^\alpha $$

with x^3 a coordinate on S^1 (i.e. with $x^3 = 0$ and $x^3 = 2\pi$ identified) and $A \equiv A_\alpha dx^\alpha$ a locally defined 1-form on $\Sigma \times R$.

2.2 Twist potential.

2.2.1 Definition.

The curvature of the connection locally represented by A is a 2-form F on $\Sigma \times R$, given by, if the equations ${}^{(4)}R_{\alpha 3}$=0 are satisfied,

$$ F_{\alpha\beta} = (1/2)e^{-4\gamma}\eta_{\alpha\beta\lambda}E^\lambda $$

with η the volume form of the metric ${}^{(3)}g$, and E an arbitrary closed 1-form . Hence if Σ is compact

$$ E = d\omega + H $$

where ω is a scalar function on V, called the twist potential, and H a representative of the 1-cohomology class of $\Sigma \times R$, for instance defined by a 1-form on Σ, harmonic for some given riemannian metric m. For simplicity we take $H = 0$.

2.2.2 Construction of A.

The connection 1- form θ can be constructed, when F is known if the following integrability condition is satisfied (see [M1], [CB-M3])

$$\int_{\Sigma_t} F = \frac{1}{2}\int_{\Sigma_t} F_{ab}dx^a \wedge dx^b = -\int_{\Sigma_t} e^{-4\gamma}N^{-1}\partial_0\omega\mu_g = 2\pi n. \tag{2.1}$$

where n is the Chern number of the bundle over $\Sigma \times R$.

We will suppose here that this bundle is trivial, i.e. $n = 0$; this value of n is the only one compatible with the smallness assumptions on the energy that we will make. The 1 - form $A \equiv A_\alpha dx^\alpha$ is then defined globally on $\Sigma \times R$. It satisfies the equation

$$dA = F \tag{2.2}$$

We denote by $\tilde{A}$ and $\tilde{F}$ the t dependent 1 - form and 2 - form on Σ given by

$$\tilde{A} \equiv A_a dx^a, \qquad \tilde{F} \equiv \frac{1}{2}F_{ab}dx^a \wedge dx^b. \tag{2.3}$$

The equation 2.2 splits into

$$d\tilde{A} = \tilde{F}, \qquad \text{i.e.} \quad \partial_a A_b - \partial_b A_a = F_{ab} \tag{2.4}$$

and, denoting by $F_{(t)}$ the 1-form $F_{ta}dx^a$:

$$\partial_t A_a - \partial_a A_t = F_{ta}, \quad \text{i.e.} \quad \partial_t \tilde{A} - dA_t = F_{(t)}. \tag{2.5}$$

We solve 2.4 by introducing a smooth m metric on Σ. We denote by δ_m and $\tilde{\Delta}_m \equiv \delta_m d + d\delta_m$ the codifferential and the de Rham Laplace operator in this metric. If we suppose that $\tilde{A}$ satisfies the Coulomb gauge condition:

$$\delta_m \tilde{A} = 0. \tag{2.6}$$

The equations 2.4 and 2.6 imply that

$$\tilde{\Delta}_m \tilde{A} = \delta_m \tilde{F}. \tag{2.7}$$

The general solution of this equation is the sum of the unique solution $\hat{A}$ which is $L^2(m)-$ orthogonal to the elements $H_{(i)}$ of a basis of harmonic 1-forms, and an arbitrary harmonic 1-form, that is:

$$\tilde{A} = \hat{A} + \sum_i c_i H_{(i)}, \quad (\hat{A}, H_{(i)})_{L^2(m)} = 0 \tag{2.8}$$

where the c_i are t-dependent numbers. The solution $\hat{A}$ satisfies a Sobolev inequality

$$||\hat{A}||_{H_2(m)} \leq C_m||\delta_m\tilde{F}||_{L^2(m)}, \quad ||\delta_m\tilde{F}||_{L^2(m)} \leq C_m||\tilde{F}||_{H_1(m)} \tag{2.9}$$

A solution $\tilde{A}$ of 2.7 satisfies 2.4 and 2.6 because 2.7 implies

$$d\tilde{\Delta}_m\tilde{A} \equiv \tilde{\Delta}_m d\tilde{A} = d\delta_m\tilde{F} \equiv \tilde{\Delta}_m\tilde{F} \tag{2.10}$$

since $\tilde{F}$ is closed, and

$$\delta_m\tilde{\Delta}_m\tilde{A} \equiv \tilde{\Delta}_m\delta_m\tilde{A} = 0. \tag{2.11}$$

2.10 implies that $d\tilde{A} - \tilde{F}$ is a harmonic 2-form on Σ, it is zero because its period, i.e. its integral on the unique 2 cycle Σ, is zero (equation 2.1). 2.11 implies that the scalar function $\delta_m\tilde{A}$ is harmonic, but its integral on Σ is zero, since it is a divergence, therefore $\delta_m\tilde{A} = 0$.

The equation 2.5 can be satisfied by choice of A_t, a t dependent function on Σ, if the 1-form $\partial_t A - F_{(t)}$ is an exact differential. The commutation of partial derivatives and the closure of F show that a solution $\tilde{A}$ of 2.4 satisfies the equation

$$\partial_a\partial_t A_b - \partial_b\partial_t A_a = \partial_t F_{ab} = \partial_a F_{tb} - \partial_b F_{ta}, \quad \text{i.e.} \quad d(\partial_t\tilde{A} - F_{(t)}) = 0. \tag{2.12}$$

Since the form $\partial_t\tilde{A} - F_{(t)}$ is closed it will be the differential of a function A_t if and only if it is $L^2(m)$ orthogonal to the harmonic 1-forms, that is, because $\partial_t\hat{A}$ is like $\hat{A}$ $L^2(m)$ orthogonal to the $H_{(i)}$ which do not depend on t :

$$(\sum_j \frac{dc_j}{dt}H_{(j)} - F_{(t)}, H_{(i)})_{L^2(m)} = 0. \tag{2.13}$$

Choosing the $H'_{(i)}s$ to be $L^2(m)$ orthonormal, this equation reduces to:

$$\frac{dc_i}{dt} = (F_{(t)}, H_{(i)})_{L^2(m)}. \tag{2.14}$$

These equations determine c_i by integration on t through its its initial value $c_i(t_0)$

We complete the determination of A_t by remarking that for a scalar function the equation 2.5 implies, using 2.6,

$$\Delta_m A_t = -\delta_m F_{(t)}, \tag{2.15}$$

an equation which determines uniquely A_t if we impose that its integral on Σ_t is zero. It satisfies then the inequality

$$||A_t||_{H_2(m)} \leq C_m||\delta_m F_{(t)}||_{L^2(m)}, \quad ||\delta_m F_{(t)}||_{L^2(m)} \leq C_m||F_t||_{H_1(m)}. \qquad (2.16)$$

Remark 2.1 *We can, instead of the Coulomb gauge, determine $\tilde{A}$ in temporal gauge, i.e. impose $A_t = 0$. We determine the 1 - form $\tilde{A}_0$, value of $\tilde{A}$ for $t = t_0$ by the relation 2.4 through the value of $\tilde{F}_0$ as above. The equation 2.5 is, when A_t and $F_{(t)}$ are known, an ordinary differential equation for $\tilde{A}$ which can be solved by integration on t :*

$$\tilde{A} = \tilde{A}_0 + \int_{t_0}^{t} F_{(t)}. \qquad (2.17)$$

When 2.5 is satisfied it implies, whatever be A_t, using the commutation of partial derivatives and the closure of F the equation

$$\partial_t(\partial_a A_b - \partial_b A_a) = \partial_a F_{tb} - \partial_b F_{ta} = \partial_t F_{ab} \qquad (2.18)$$

i.e.

$$\partial_t(d\tilde{A} - \tilde{F}) = 0 \qquad (2.19)$$

hence $d\tilde{A} - \tilde{F} = 0$ for all t if it is so for $t = t_0$.

The disavantage of the temporal gauge is that it gives only H_1 estimates for $\tilde{A}$.

2.3 Wave map equation.

The fact that F is a closed form together with the equation ${}^{(4)}R_{33} = 0$ imply (with the choice $H = 0$ in the definition of ω) that the pair $u \equiv (\gamma, \omega)$ satisfies a wave map equation from $(\Sigma \times R,{}^{(3)} g)$ into the Poincaré plane (R^2, G),

$$G = 2(d\gamma)^2 + (1/2)e^{-4\gamma}(d\omega)^2,$$

It is a system of hyperbolic type when ${}^{(3)}g$ is a known lorentzian metric which reads, denoting by ${}^{(3)}\nabla_\alpha$ the components of covariant derivatives of tensors on $\Sigma \times R$ in the metric ${}^{(3)}g$ in the moving frame (θ^a, dt):

$${}^{(3)}\nabla^\alpha \partial_\alpha \gamma + \frac{1}{2} e^{-4\gamma} g^{\alpha\beta} \partial_\alpha \omega \partial_\beta \omega = 0$$

$$^{(3)}\nabla^\alpha \partial_\alpha \omega - 4 g^{\alpha\beta} \partial_\alpha \omega \partial_\beta \gamma = 0$$

The integral 2.1 is independent of t if F is closed, hence if the wave map equation is satisfied.

Remark 2.2 *The non zero Christoffel symbols of the metric G are*

$$G^1_{22} \equiv G^\gamma_{\omega\omega} = \frac{1}{2} e^{-4\gamma}, \quad G^2_{12} = G^2_{21} = G^\omega_{\gamma\omega} = -2$$

The scalar and Riemann curvature are:

$$R_{12,12} = -2e^{-4\gamma}, \quad R = -4$$

2.4 3-dimensional Einstein equations

When $^{(4)}R_{3\alpha} = 0$ and $^{(4)}R_{33} = 0$ the Einstein equations $^{(4)}R_{\alpha\beta} = 0$ are equivalent to Einstein equations on the 3-manifold $\Sigma \times R$ for the metric $^{(3)}g$ with source the stress energy tensor of the wave map:

$$^{(3)}R_{\alpha\beta} = \rho_{\alpha\beta} \equiv \partial_\alpha u . \partial_\beta u$$

where a dot denotes a scalar product in the metric of the Poincaré plane:

$$\partial_\alpha u . \partial_\beta u \equiv 2 \partial_\alpha \gamma \partial_\beta \gamma + \frac{1}{2} e^{-4\gamma} \partial_\alpha \omega \partial_\beta \omega$$

In dimension 3 the Einstein equations are non dynamical, except for the conformal class of g determined by Teichmüller parameters. They decompose into:

a. Constraints.

b. Equations for lapse and shift to be satisfied on each Σ_t. These equations, as well as the constraints, are of elliptic type.

c. Evolution equations for the Teichmüller parameters, ordinary differential equations.

2.4.1 Constraints on Σ_t.

One denotes by k the extrinsic curvature of Σ_t as submanifold of $(\Sigma \times R, ^{(3)} g)$. Then, with ∇ the covariant derivative in the metric g,

$$k_{ab} \equiv (2N)^{-1}(-\partial_t g_{ab} + \nabla_a \nu_b + \nabla_b \nu_a),$$

the equations (momentum constraint)

$$^{(3)}R_{0a} \equiv N(-\nabla_b k_a^b + \partial_a \tau) = \partial_0 u.\partial_a u \tag{2.20}$$

and (hamiltonian constraint)

$$2N^{-2(3)}S_{00} \equiv R(g) - k_b^a k_a^b + \tau^2 = N^{-2}\partial_0 u.\partial_0 u + g^{ab}\partial_a u.\partial_b u \tag{2.21}$$

do not contain second derivatives transversal to Σ_t of g or u. They are the constraints. To transform the constraints into an elliptic system one uses the conformal method. We set

$$g_{ab} = e^{2\lambda}\sigma_{ab},$$

where σ is a riemannian metric on Σ, depending on t, on which we will comment later, and

$$k_{ab} = h_{ab} + \frac{1}{2}g_{ab}\tau$$

where τ is the g-trace of k, hence h is traceless.

We denote by D a covariant derivation in the metric σ. We set

$$u' = N^{-1}\partial_0 u$$

with ∂_0 the Pfaff derivative of u, namely

$$\partial_0 = \frac{\partial}{\partial t} - \nu^a \partial_a \text{ with } \partial_a = \frac{\partial}{\partial x^a}$$

and

$$\dot{u} = e^{2\lambda}u'$$

The momentum constraint on Σ_t reads if τ is constant in space, a choice which we will make

$$D_b h_a^b = L_a, L_a \equiv -D_a u.\dot{u} \tag{2.22}$$

This is a linear equation for h, with left hand side independent of λ. The general solution is the sum of a transverse traceless tensor $h_{TT} \equiv q$ (see 2.28 below) and a conformal Lie derivative r. Such tensors are L^2-orthogonal on (Σ, σ).

The hamiltonian constraint reads as the semilinear elliptic equation in λ :

$$\Delta\lambda = f(x,\lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3, \tag{2.23}$$

with $\Delta \equiv \Delta_\sigma$ the Laplacian in the metric σ and:

$$p_1 \equiv \frac{1}{4}\tau^2, \quad p_2 \equiv \frac{1}{2}(|\ \dot{u}\ |^2 + |\ h\ |^2), \quad p_3 \equiv \frac{1}{2}(R(\sigma) - |Du|^2)$$

2.4.2 Equations for lapse and shift.

Lapse and shift are gauge parameters for which we obtain elliptic equations on each Σ_t as follows.

We impose that the $\Sigma_t's$ have constant (in space) mean curvature, namely that τ is a given increasing function of t. The lapse N satisfies then the linear elliptic equation

$$\Delta N - \alpha N = -e^{2\lambda}\frac{\partial \tau}{\partial t} \tag{2.24}$$

with ($|.|$pointwise norm in the metric σ)

$$\alpha \equiv e^{-2\lambda}(|\, h\,|^2 + |\, \dot{u}\,|^2) + \frac{1}{2}e^{2\lambda}\tau^2 \tag{2.25}$$

The equation to be satisfied by the shift ν results from the the expression for h deduced from the definition of k

$$h_{ab} \equiv (2N)^{-1}[-(\partial_t g_{ab} - \frac{1}{2}g_{ab}g^{cd}\partial_t g_{cd}) + \nabla_a \nu_b + \nabla_b \nu_a - g_{ab}\nabla_c \nu^c]$$

which implies, if $g_{ab} \equiv e^{2\lambda}\sigma_{ab}$ and if $n_a \equiv e^{-2\lambda}\nu_a$ denotes the covariant components of the shift vector ν in the metric σ (thus $n^a = \nu^a$)

$$h_{ab} \equiv (2N)^{-1}e^{2\lambda}[-(\partial_t \sigma_{ab} - \frac{1}{2}\sigma_{ab}\sigma^{cd}\partial_t \sigma_{cd}) + D_a n_b + D_b n_a - \sigma_{ab}D_c n^c]$$

The equation is therefore:

$$(L_\sigma n)_{ab} \equiv D_a n_b + D_b n_a - \sigma_{ab}D_c n^c = f_{ab} \tag{2.26}$$

$$f_{ab} \equiv 2Ne^{-2\lambda}h_{ab} + \partial_t \sigma_{ab} - \frac{1}{2}\sigma_{ab}\sigma^{cd}\partial_t \sigma_{cd} \tag{2.27}$$

The homogeneous associated operator, the conformal Killing operator L_σ, has injective symbol, and it has a kernel zero, since manifolds of genus greater than 1 admit no conformal Killing fields.

The kernel of the dual of L_σ is the space of transverse traceless symmetric 2-tensors, i.e. symmetric 2-tensors T such that

$$\sigma^{ab}T_{ab} = 0, \quad D^a T_{ab} = 0. \tag{2.28}$$

These tensors are usually called TT tensors. The spaces of TT tensors are the same for two conformal metrics.

2.5 Teichmüller parameters.

On a compact 2-dimensional manifold of genus G $\geq$ 2 the space T_{eich} of conformally inequivalent riemannian metrics, called Teichmüller space, can be identified (cf. Fisher and Tromba) with M_{-1}/D_0, the quotient of the space of metrics with scalar curvature -1 by the group of diffeomorphisms homotopic to the identity. $M_{-1} \rightarrow T_{eich}$ is a trivial fiber bundle whose base can be endowed with the structure of the manifold R^n, with $n = 6G - 6$.

We require the metric σ_t to be in some chosen cross section $Q \rightarrow \psi(Q)$ of the above fiber bundle. Let $Q^I, I = 1, ..., n$ be coordinates in T_{eich}, then $\partial\psi/\partial Q^I$ is a known tangent vector to M_{-1} at $\psi(Q)$, that is a symmetric 2-tensor field on Σ, the sum of a transverse traceless tensor field $X_I(Q)$ and of the Lie derivative of a vector field on the manifold $(\Sigma, \psi(Q))$. The tensor fields $X_I(Q), I = 1, ...n$ span the space of transverse traceless tensor fields on $(\Sigma, \psi(Q))$. The matrix with elements

$$\int_\Sigma X_I^{ab} X_{Jab} \mu_{\psi(Q)}$$

is invertible.

We have found in [CB-M1] an ordinary differential system satisfied by $t \mapsto Q(t)$ by using on the one hand the solvability condition for the shift equation which determines dQ^I/dt in terms of h_t which reads

$$\int_{\Sigma_t} f_{ab} X_J^{ab} \mu_{\sigma_t} = 0, \quad J = 1, ...6G - 6, \tag{2.29}$$

and on the other hand the necessary and sufficient conditions for the previous equations to imply also the remaining equations ${}^{(3)}R_{ab} - \rho_{ab} = 0$, that is:

$$\int_{\Sigma_t} N({}^{(3)}R_{ab} - \rho_{ab}) X_J^{ab} \mu_{\sigma_t} = 0, \text{ for } J = 1, 2, ...6G - 6. \tag{2.30}$$

We have used the expression

$$\partial_t \sigma_{ab} = \frac{dQ^I}{dt} X_{I,ab} + C_{ab}$$

where C_{ab} is a Lie derivative, L^2 orthogonal to TT tensors, together with the decomposition $h = q + r$, with r a tensor in the range of the conformal

Killing operator and q a TT tensor. This last tensor can be written with the use of the basis X_I of such tensors, the coefficients P^I depending only on t:

$$q_{ab} = P^I(t) X_{I,ab}$$

The orthogonality condition 2.29 reads, using the expression 2.27 of f_{ab} and the fact that the transverse tensors X_I are orthogonal to Lie derivatives and are traceless:

$$\int_{\Sigma_t} [2Ne^{-2\lambda}(r_{ab} + P^I X_{I,ab}) + (dQ^I/dt) X_{I,ab}] X_J^{ab} \mu_\sigma = 0$$

The tangent vector dQ^I/dt to the curve $t \to Q(t)$ and the tangent vector $P^I(t)$ to T_{eich} are therefore linked by the linear system

$$X_{IJ} \frac{dQ^I}{dt} + Y_{IJ} P^I + Z_J = 0$$

with

$$X_{IJ} \equiv \int_{\Sigma_t} X_I^{ab} X_{J,ab} \mu_\sigma, \tag{2.31}$$

$$Y_{IJ} \equiv \int_{\Sigma_t} 2Ne^{-2\lambda} X_I^{ab} X_{J,ab} \mu_\sigma, \quad Z_J \equiv \int_{\Sigma_t} 2Ne^{-2\lambda} r_{ab} X_J^{ab} \mu_\sigma \tag{2.32}$$

While, using

$${}^{(3)}R_{ab} \equiv R_{ab} - N^{-1} \bar{\partial}_0 k_{ab} - 2k_{ac} k_b^c + \tau k_{ab} - N^{-1} \nabla_a \partial_b N \tag{2.33}$$

where

$$R_{ab} \equiv \frac{1}{2} R g_{ab}, \quad \rho_{ab} \equiv \partial_a u . \partial_b u$$

$$k_{ab} \equiv P^I X_{I,ab} + r_{ab} + \frac{1}{2} g_{ab} \tau$$

and $\bar{\partial}_0$ is an operator on time dependent space tensors defined by, with $\mathcal{L}_\nu$ the Lie derivative in the direction of ν,

$$\bar{\partial}_0 \equiv \partial_t - \mathcal{L}_\nu$$

gives[1] for 2.30 the expression:

$$\int_{\Sigma_t} (-\bar{\partial}_0 k_{ab} - 2Ne^{2\lambda} h_{ac} h^c_b + \tau N h_{ab} - \nabla_a \partial_b N - \partial_a u.\partial_b u) X^{ab}_J \mu_{\sigma_t} = 0 \quad (2.34)$$

We have thus obtained an ordinary differential system of the form

$$X_{IJ} \frac{dP^I}{dt} + \Phi_J(P, \frac{dQ}{dt}) = 0$$

where Φ is a polynomial of degree 2 in P and dQ/dt with coefficients depending smoothly on Q and directly but continuously on t through the other unknowns, namely:

$$\Phi_J \equiv A_{JIK} P^I P^K + B_{JIK} P^I \frac{dQ^K}{dt} + C_{JI} P^I + D_J$$

with

$$A_{JIK} \equiv \int_{\Sigma_t} 2Ne^{2\lambda} X^c_{I,a} X_{K,bc} X^{ab}_J \mu_{\sigma_t}$$

$$B_{JIK} \equiv \int_{\Sigma_t} \frac{\partial X_{I,ab}}{\partial Q^K} X^{ab}_J \mu_{\sigma_t}$$

$$C_{JI} \equiv \int_{\Sigma_t} [(-\mathcal{L}_\nu X_I)_{ab} + 4Ne^{-2\lambda} r^c_b X_{I,ac} - \tau N X_{I,ab}] X^{ab}_J \mu_{\sigma_t}$$

and, using integration by parts and the transverse property of the X_I to eliminate second derivatives of N (recall that $\nabla_a \partial_b N \equiv D_a \partial_b N - 2\partial_a \lambda \partial_b N$)

$$D_J \equiv \int_{\Sigma_t} (-\bar{\partial}_0 r_{ab} - 2Ne^{-2\lambda} r_{ac} r^c_b + \tau N r_{ab} + 2\partial_a \lambda \partial_b N - \partial_a u.\partial_b u) X^{ab}_J \mu_{\sigma_t}.$$

3 Cauchy problem.

3.1 Cauchy data.

The Cauchy data on Σ_{t_0} are:

1. A C^∞ riemannian metric σ_0 which projects onto a point $Q(t_0)$ of T_{eich} and a C^∞ tensor q_0 which is TT in the metric σ_0.

[1] In the formula 2.33 indices are raised with g, in 2.34 they are raised with σ.

2. Cauchy data for u and $\dot{u}$ on Σ_{t_0}, i.e.

$$u(t_0, .) = u_0, \quad \dot{u}(t_0, .) = \dot{u}_0.$$

We say that a pair of scalar functions, $u \equiv (\gamma, \omega)$ or $\dot{u} \equiv (\dot{\gamma}, \dot{\omega})$ belongs to W^p_s if it is so of each of the scalars; W^p_s and $H_s \equiv W^2_s$ are the usual Sobolev spaces of scalar functions on the riemannian manifold (Σ, σ_0). We suppose that

$$u_0 \in H_2, \quad \dot{u}_0 \in H_1.$$

¿From these data one determines the values on Σ_0 of the auxiliary unkown, $h_0 \in W^p_2$, $1 < p < 2$, the conformal factor lapse and shift λ_0, N_0, $\nu_0 \in W^p_3$.

One deduces then the usual Cauchy data for the wave map by

$$(\partial_t u)_0 = e^{-2\lambda_0} N_0 \dot{u}_0 + \nu_0^a \partial_a u_0 \tag{3.1}$$

It holds that

$$(\partial_t u)_0 \in H_1. \tag{3.2}$$

We suppose that the initial data satisfy the integrability condition 2.1 and we deduce from them an admissible $\tilde{A}_0$.

3.2 Local in time existence theorem.

The following theorem is a consequence of previous results (see CB-M2, CB1, CB-M1).

Theorem 3.1 *The Cauchy problem with the above data for the Einstein equations with S^1 isometry group has, if $T - t_0$ is small enough, a solution with $u \in C^0([t_0, T), H_2)$ $\dot{u} \in C^1([t_0, T), H_1)$; $\lambda, N, \nu \in C^0([t_0, T), W^p_3)$ $\cap C^1([t_0, T), W^p_2)$, $1 < p < 2$ and $N > 0$ while $\sigma \in C^1([t_0, T), C^\infty)$ with σ_t uniformly equivalent to σ_0. This solution is unique up to t parametrization of τ, choice of A_t, and choice of a cross section of M_{-1} over T_{eich}.*

3.3 Scheme for global existence.

If the universe is expanding the mean curvature τ starts negative and increases, the universe attains a moment of maximum expansion if it exists up to $\tau = 0$. We choose the time parameter t by requiring that

$$t = -\frac{1}{\tau}. \tag{3.3}$$

Then t increases from $t_0 > 0$ to infinity when τ increases from $\tau_0 < 0$ to zero.

It results from the local existence theorem and a standard argument that the solution of the Einstein equations exists on $[t_0, \infty)$ if the curve $t \mapsto Q(t)$ remains in a compact subset of T_{eich} and the norms $||\gamma(t,.), \omega(t,.)||_{H_2}$, $||\partial_t\gamma(t,.), \partial_t\omega(t,.)||_{H_1}$ do not blow up for any finite t.

It will result from the following sections that these norms do not blow up if it is so of the energies that we will now define. However this non blow up will be proved only for small initial data and the proof of the boundedness of Q will require the consideration of corrected energies, analogous to the corrected energies introduced in [CB-M1], but linked with the wave structure and more complicated to estimate.

4 Energies.

4.1 First energy.

4.1.1 Definition.

We denote by $|.|$ a norm in the metric G and $|.|_g$ a norm in the metrics g and G, in particular:

$$|u'|^2 \equiv 2(\gamma')^2 + \frac{1}{2}e^{-4\gamma}(\omega')^2 \ , \quad |Du|^2_g \equiv g^{ab}(2D_a\gamma D_b\gamma + \frac{1}{2}e^{-4\gamma}D_a\omega D_b\omega) \tag{4.1}$$

The 2+1 dimensional Einstein equations with source the stress energy tensor of the wave map u contain the following equation (hamiltonian constraint)

$$2N^{-2}(T_{00} -^{(3)} S_{00}) = |u'|^2 + |Du|^2_g + |k|^2_g - R(g) - \tau^2 = 0 \tag{4.2}$$

The splitting of the covariant 2-tensor k into a trace and a traceless part:

$$k_{ab} = h_{ab} + \frac{1}{2}g_{ab}\tau \tag{4.3}$$

gives that:

$$|k|^2_g = g^{ac}g^{bd}k_{ab}k_{cd} = |h|^2_g + \frac{1}{2}\tau^2 \tag{4.4}$$

and the hamiltonian constraint equation reads

$$|u'|^2 + |Du|^2_g + |h|^2_g = R(g) + \frac{1}{2}\tau^2 \tag{4.5}$$

Inspired by this equation, we define the first energy by the following formula

$$E(t) \equiv \int_{\Sigma_t} (I_0 + I_1 + \frac{1}{2}|h|_g^2)\mu_g$$

with

$$I_0 \equiv \frac{1}{2} \mid u' \mid^2 \equiv (\gamma')^2 + \frac{1}{4}e^{-4\gamma}(\omega')^2,$$
$$I_1 \equiv \frac{1}{2} \mid Du \mid_g^2 \equiv \mid D\gamma \mid_g^2 + \frac{1}{4}e^{-4\gamma} \mid D\omega \mid_g^2$$

that is:

$$E(t) \equiv \frac{1}{2}\{\parallel u' \parallel_g^2 + \parallel Du \parallel_g^2 + \parallel h \parallel_g^2\} \tag{4.6}$$

with $\parallel . \parallel_g^2$the square of the integral in the metric g of $|.|_g^2$. This energy is the first energy of the wave map u completed by the square of the $L^2(g)$ norm of h.

4.2 Bound of the first energy.

The integration of the hamiltonian constraint on $(\Sigma_{t,}g)$ using the constancy of τ and the Gauss Bonnet theorem which reads, with χ the Euler characteristic of Σ

$$\int_{\Sigma_t} R(g)\mu_g = 4\pi\chi \tag{4.7}$$

shows that

$$E(t) = \frac{\tau^2}{4}Vol_g(\Sigma_t) + 2\pi\chi, \quad Vol_g(\Sigma_t) = \int_{\Sigma_t} \mu_g.$$

Recall that on a compact manifold

$$\frac{dVol_g\Sigma_t}{dt} = \frac{1}{2}\int_{\Sigma_t} g^{ab}\frac{\partial g_{ab}}{\partial t}\mu_g = -\tau \int_{\Sigma_t} N\mu_g.$$

We use the equation

$$N^{-1(3)}R_{00} \equiv \Delta_g N - N|k|_g^2 + \partial_t \tau = N|u'|^2$$

together with the splitting of k to write after integration, since τ is constant in space,

$$\frac{1}{2}\tau^2\int_{\Sigma_t} N\mu_g = \frac{d\tau}{dt}Vol_g(\Sigma_t) - \int_{\Sigma_t} N(|h|_g^2 + |u'|^2)\mu_g$$

We then find as in [CB-M1] that it simplifies to:

$$\frac{dE(t)}{dt} = \frac{1}{2}\tau\int_t (|h|_g^2 + |u'|^2)N\mu_g. \tag{4.8}$$

We see that $E(t)$ is a non increasing[2] function of t if τ is negative. We remark that, due to the use of the constraints DN does not appear in 4.8, as it would have if we had used only the wave map energy.

We set

$$\varepsilon \equiv \{E(t)\}^{\frac{1}{2}}, \quad \varepsilon_0 \equiv \{E(t_0)\}^{\frac{1}{2}}, \tag{4.9}$$

we have proved that if $\tau \leq 0$ then

$$\varepsilon \leq \varepsilon_0. \tag{4.10}$$

4.3 Second energy.

4.3.1 notations.

We denote by $\hat{\nabla}$ a covariant derivative in the metrics g and G, for t dependent sections of the fiber bundle $E^{p,q}$ with base Σ and fiber $\otimes^p T_x^*\Sigma \otimes^q T_{u(x)}P$, with P the Poincaré plane. That is we set, for $\partial_c u^A$, a section of $E^{1,1}$,

$$\hat{\nabla}_b\partial_c u^A \equiv \partial_b\partial_c u^A - \Gamma^a_{bc}\partial^A_a + G^A_{BC}\partial_b u^B\partial_c u^C \tag{4.11}$$

where Γ^a_{bc} and G^A_{BC} denote respectively the connection coefficients of the metric g, and of G given in the remark 2.2. For u'^A, section of $E^{0,1}$, we have:

$$\hat{\nabla}_a u'^A = \partial_a u'^A + G^A_{BC}\partial_a u^B u'^C. \tag{4.12}$$

while for G_{AB}, section of $E^{0,2}$ it holds that

$$\hat{\nabla}_a G_{AB} = 0. \tag{4.13}$$

[2] The absence of the term $|\mathrm{Du}|^2$ prevents the use of this equality to obtain a decay estimate.

On the other hand we define by $\hat{\partial}_0$ a differential operator mapping a t dependent section of a bundle $E^{p,q}$ into another such section by the formula:

$$\hat{\partial}_0 \hat{\nabla}^p u^A = \bar{\partial}_0 \hat{\nabla}^p u^A + G^A_{BC} \partial_0 u^B \hat{\nabla}^p u^C \tag{4.14}$$

with [3]

$$\bar{\partial}_0 \equiv \partial_0 - \mathcal{L}_\nu \tag{4.15}$$

where $\mathcal{L}_\nu$ denotes the Lie derivative with respect to the shift ν. In particular:

$$\hat{\partial}_0 u'^A = \partial_0 u'^A + G^A_{BC} \partial_0 u^B u'^C \tag{4.16}$$

and

$$\hat{\partial}_0 G^{AB} = 0. \tag{4.17}$$

With these notations the wave map equation reads:

$$-\hat{\partial}_0 u'^A + \hat{\nabla}^a (N \partial_a u^A) + N\tau u' = 0. \tag{4.18}$$

We will use the following lemma, which can be foreseen, and also checked[4] by direct computation.

Lemma 4.1 *The following commutation relations are satisfied:*

$$\hat{\partial}_0 \partial_a u^A = \hat{\nabla}_a \partial_0 u^A, \tag{4.19}$$

$$\hat{\partial}_0 \hat{\nabla}_a \partial_0 u^A - \hat{\nabla}_a \hat{\partial}_0 \partial_0 u^A = R_{CB}{}^A{}_D \partial_0 u^C \partial_a u^B \partial_0 u^D, \tag{4.20}$$

$$\hat{\partial}_0 \hat{\nabla}_a \partial_b u^A - \hat{\nabla}_a \hat{\partial}_0 \partial_b u^A = R_{CB}{}^A{}_D \partial_0 u^C \partial_a u^B \partial_b u^D - \partial_c u^A \hat{\partial}_0 \Gamma^c_{ab}. \tag{4.21}$$

We recall the identities

$$\hat{\partial}_0 g^{ab} = \bar{\partial}_0 g^{ab} = 2Nk^{ab}, \tag{4.22}$$

$$\hat{\partial}_0 \Gamma^c_{ab} = \bar{\partial}_0 \Gamma^c_{ab} = \nabla^c (N k_{ab}) - \nabla_a (N k^c_b) - \nabla_b (N k^c_a). \tag{4.23}$$

[3] Operator on tensors denoted $\hat{\partial}_0$ in [CB-Y].

[4] Used in the case of tensor fields in [CB-Y].

4.3.2 Definition.

We define the second energy by the following formula

$$E^{(1)}(t) \equiv \int_{\Sigma_t} (J_0 + J_1)\mu_g \tag{4.24}$$

with

$$J_1 = \frac{1}{2} \mid \hat{\Delta}_g u \mid^2 \equiv \frac{1}{2}\{(2(\hat{\Delta}_g \gamma)^2 + \frac{1}{2}e^{-4\gamma}(\hat{\Delta}_g \omega)^2\} \tag{4.25}$$

$$J_0 = \frac{1}{2} \mid \hat{\nabla} u' \mid^2_{g,} \equiv \frac{1}{2}\{2|\hat{\nabla}\gamma'|^2_g + \frac{1}{2}e^{-4\gamma}|\hat{\nabla}\omega'|^2_g\}. \tag{4.26}$$

4.3.3 Estimate.

We postpone the computation and estimate of the derivative of $E^{(1)}(t)$ until after the estimates of h, λ, N and ν. We set:

$$E(t) \equiv \varepsilon^2, \quad E^{(1)}(t) \equiv \tau^2 \varepsilon_1^2 \tag{4.27}$$

4.4 Norms.

We suppose chosen a smooth cross section $Q \to \psi(Q)$ of M_{-1} over the Teichmuller space T_{eich}, together with a C^1 curve $t \to Q(t)$. We are then given by lift to M_{-1} a regular metric σ_t for $t \in [t_0, T]$, with scalar curvature -1.

Definition 4.1 *Hypothesis H_σ : the curve is contained in a compact subset of T_{eich}.*

Under the hypothesis H_σ the metric σ_t is uniformly equivalent to the metric $\sigma_0 \equiv \sigma_{t_0}$. A $t-$ dependent Sobolev constant on (Σ, σ_t) is uniformly equivalent to a number. We denote by C_σ any such number, which depends only on the considered compact subset of T_{eich}.

The spaces $W^p_s(\sigma_t)$ are the usual Sobolev spaces of tensor fields on the riemannian manifold (Σ, σ_t). By the hypothesis on σ_t their norms are uniformly equivalent for $t \in [t_{0,}, T]$ to the norm in $W^p_s(\sigma_0)$ denoted simply W^p_s. We set $W^2_s = H_s$.

We have denoted by $|.|$ a pointwise norm in the σ and G metrics; $\|\ .\ \|$ and $\|\ .\ \|_p$ denote L^2 and L^p norms in the σ metric.

We denote by $\hat{D}$ a covariant derivative relative to the metrics σ and G.

A lower case index m or M denotes respectively the lower or upper bound of a scalar function on Σ_t. It may depend on t.

Lemma 4.2 *It holds that:*

1.

$$||Du||^2 \equiv ||Du||_g^2 \leq 2\varepsilon^2, \quad ||u'||^2 \leq e^{-2\lambda_m}||u'||_g^2 \leq 2e^{-2\lambda_m}\varepsilon^2. \tag{4.28}$$

2.

$$||\hat{D}Du||^2 \leq 2e^{2\lambda_M}\tau^2\varepsilon_1^2 + \varepsilon^2. \tag{4.29}$$

PROOF. 1. results directly from the definitions.

2. By definition

$$||\hat{D}Du||^2 = \int_\Sigma \hat{D}^a D^b u.\hat{D}_a D_b u \mu_\sigma \tag{4.30}$$

$$= \int_\Sigma \{\hat{D}^a(D^b u.\hat{D}_a D_b u) - D^b u.\hat{D}^a \hat{D}_a D_b u\}\mu_\sigma = -\int_{\Sigma_t} D^b u.\hat{D}^a \hat{D}_a D_b u \mu_\sigma \tag{4.31}$$

(since $D^b u.\hat{\nabla}_a D_b u$ is an ordinary covariant vector on Σ its divergence integrates to zero).

The Ricci commutation formula gives that, with $\rho_{ab} = -\frac{1}{2}\sigma_{ab}$ the Ricci curvature of the metric σ:

$$\hat{D}^a \hat{D}_a D_b u^C = \hat{D}^a \hat{D}_b D_a u^C = \hat{D}_b \hat{\Delta} u^C + \rho_b{}^c D_c u^C + D^a u^A D_b u^B R_{AB,}{}^C{}_D D_a u^D. \tag{4.32}$$

By another integration by parts 4.31 gives then

$$\int_\Sigma -D^b u.\hat{D}^a \hat{D}_a D_b u \mu_\sigma = \int_\Sigma \{|\hat{\Delta} u|^2 - D^b u.(\rho_b^c D_c u^C + D^a u^A D_b u^B R_{AB,}{}^{\cdot}{}_D D_a u^D)\}\mu_\sigma$$

On a 2 dimensional manifold the Riemann curvature is given by:

$$R_{AB,}{}^C{}_D = \frac{1}{2}R(G)\{\delta_A^C G_{BD} - \delta_B^C G_{AD}\} \tag{4.33}$$

A straightforward computation gives therefore

$$D^b u.D^a u^A D_b u^B R_{AB,}{}^{\cdot}{}_D D_a u^D = \tag{4.34}$$

$$\frac{1}{2}R(G) D^b u^E D^a u^A D_b u^B D_a u^D \{G_{AE}G_{BD} - G_{BE}G_{AD}\} = \tag{4.35}$$

$$\frac{1}{2}R(G)\{(D^b u.D^a u)(D_b u.D_a u) - (D^b u.D_b u)(D^a u.D_a u)\} = \tag{4.36}$$

$$\frac{1}{2}R(G)\{|Du.Du|^2 - |\ |Du|^2|^2\} \tag{4.37}$$

In the case of the Poincaré plane $R(G) = -4$, hence

$$-D^b u.D^a u^A D_b u^B R_{AB,}{}^{\cdot}{}_D D_a u^D = 2\{|Du.Du|^2 - |\ |Du|^2|^2\} \leq 0, \tag{4.38}$$

because

$$|Du.Du| \leq \ |Du|^2. \tag{4.39}$$

■

5 First elliptic estimates.

The equations for h, λ, N, and ν are elliptic equations on (Σ_t, σ_t), identical with those written in [CB-M1], except that in the coefficients $Du.\dot{u}$, $|Du|^2$, $|\dot{u}|^2$ which appear in these equations u is now a wave map and not a scalar function. The estimates obtained in [1] in terms of ε and ε_1 will be valid if the new coefficients satisfy the same estimates in terms of our new ε and ε_1.

5.1 Basic bounds on N and λ.

The generalized maximum principle[5] applied to the equations 2.24 and 2.23 satisfied respectively by N and λ shows that, with our choice of the time parameter

$$t = -\tau^{-1}, \tag{5.1}$$

it holds that

$$0 \leq N_m \leq N \leq N_M \leq 2, \tag{5.2}$$

$$e^{-2\lambda_M} \leq e^{-2\lambda} \leq e^{-2\lambda_m} \leq \frac{1}{2}\tau^2. \tag{5.3}$$

Definition 5.1 *We say that the hypothesis[6] H_λ is satisfied if there exists a number $c_\lambda > 1$, independent of t, such that*

$$\frac{1}{\sqrt{2}}e^{\lambda_M}|\tau| \leq c_\lambda. \tag{5.4}$$

and we denote by C_λ any positive continuous function of $c_\lambda \in R^+$.

[5] The coefficients in these equations belong to the same functional spaces as in [CB-M1], as will be proved in the next subsection which will also estimate them.

[6] This hypothesis replaces the hypothesis H_c made on v in [CB-M1].

5.2 L^2 estimates of $|| \, |Du|^2||$ and $|| \, |\dot{u}|^2||$.

Under the hypothesis H_σ and H_λ there exist numbers C_σ, C_λ such that u satisfies the same inequalities as in the polarized case, that is:

Lemma 5.1

$$|| \, |Du|^2|| \leq C_\sigma C_\lambda \{\varepsilon^2 + \varepsilon\varepsilon_1\}. \tag{5.5}$$

$$\| \mid u' \mid^2 \| \leq C_\sigma C_\lambda \tau^2 \varepsilon(\varepsilon + \varepsilon_1). \tag{5.6}$$

$$\| \mid \dot{u} \mid^2 \| \leq C_\sigma C_\lambda \tau^{-2} \varepsilon(\varepsilon + \varepsilon_1). \tag{5.7}$$

Proof. A Sobolev embedding theorem applied to the scalar function $|u'|^2$ gives that:[7]

$$\| \mid u' \mid^2 \| \leq C_\sigma (\| |u'|^2 \|_1 + \|D|u'|^2\|_1).$$

It holds that

$$D|u'|^2 \equiv 2u^{'}.\hat{D}u'$$

therefore, since $\hat{D}u' \equiv \hat{\nabla}u'$,

$$\|D|u'|^2\|_1 \leq 2||u'|| \; ||\hat{D}u'|| \leq 2e^{-\lambda_m}||u'||_g \; ||\hat{\nabla}u'||_g$$

Hence, since $\||u'|^2\|_1 \equiv ||u'||^2$, and using the bound 5.3 of λ

$$\| \mid u' \mid^2 \| \leq C_\sigma e^{-\lambda_m}||u'||_g(e^{-\lambda_m}||u'||_g + ||\hat{D}u'||_g) \leq C_\sigma |\tau|^2 \varepsilon(\varepsilon + \varepsilon_1)$$

By the definition of $\dot{u}$ it holds that.

$$\| \, |\dot{u}|^2 \, \| \leq e^{4\lambda_M} \, \| \, |u'|^2 \, \|,$$

hence

$$\| \mid \dot{u} \mid^2 \| \leq C_\sigma e^{4\lambda_M} \tau^2 \varepsilon(\varepsilon + e^{\lambda_M}|\tau|\varepsilon_1) \leq C_\sigma C_\lambda \tau^{-2} \varepsilon(\varepsilon + \varepsilon_1).$$

On the other hand, since

$$D|Du|^2 \equiv 2Du.\hat{D}Du, \tag{5.8}$$

it holds that, using again the Sobolev embedding theorem

$$\| \mid Du \mid^2 \| \leq C_\sigma \|Du\|(\|Du\| + \|\hat{D}Du\|)$$

[7]See C.B 98 p. 160., here case n=2.

which gives using lemma 4.2

$$\| \mid Du \mid^2 \| \leq C_\sigma \varepsilon(\varepsilon + e^{\lambda_M}|\tau|\varepsilon_1)$$

hence under the hypothesis $\mathrm{H}_{\lambda:}$

$$\| \mid Du \mid^2 \| \leq C_\sigma C_\lambda \varepsilon(\varepsilon + \varepsilon_1).$$

■

Lemma 5.2

$$|| \, |Du|^2||_g \leq C_\sigma C_\lambda |\tau| \varepsilon\{\varepsilon + \varepsilon_1\}. \tag{5.9}$$

$$\| \mid u' \mid^2 \|_g \leq C_\sigma C_\lambda |\tau| \varepsilon(\varepsilon + \varepsilon_1). \tag{5.10}$$

PROOF. The inequalities 5.5 and 5.6 imply that

$$\| \mid u' \mid^2 \|_g \equiv (\int_{\Sigma_t} |u'|^4 \mu_g)^{\frac{1}{2}} \leq e^{\lambda_M} \| \mid u' \mid^2 \| \leq C_\lambda C_\sigma |\tau| \varepsilon(\varepsilon + \varepsilon_1)$$

and, using the lower bound of λ,

$$\| \mid Du \mid^2 \|_g \equiv (\int_{\Sigma_t} |Du|_g^4 \mu_g)^{\frac{1}{2}} \leq e^{-\lambda_m} \| \mid Du \mid^2 \| \leq C_\sigma C_\lambda |\tau| \varepsilon(\varepsilon + \varepsilon_1).$$

■

5.3 Estimate of h in H_1.

We have defined the auxiliary unknown h by

$$h_{ab} \equiv k_{ab} - \frac{1}{2} g_{ab} \tau$$

5.3.1 Estimate of $||h||$.

The L^2 norm of h on (Σ, σ) is bounded in terms of the first energy and an upper bound λ_M of the conformal factor since we have

$$\| h \|^2 = \int_{\Sigma_t} \sigma^{ac} \sigma^{bd} h_{ab} h_{cd} \mu_\sigma \leq e^{2\lambda_M} \| h \|^2_{L^2(g)} \leq 2 e^{2\lambda_M} \varepsilon^2.$$

5.3.2 Estimate of $\| Dh \|$.

The tensor h satisfies the equations

$$D_a h^a_b = L_b \equiv -\partial_a u.\dot{u}$$

It is the sum of a TT (transverse, traceless) tensor $h_{TT} \equiv q$ and a conformal Lie derivative r:

$$h \equiv q + r$$

It results from elliptic theory that on each Σ_t the tensor r satisfies the estimate

$$\| r \|_{H_1} \leq C_\sigma \| Du.\dot{u} \| \leq C_\sigma \| |Du|^2 \|^{\frac{1}{2}} \| |\dot{u}|^2 \|^{\frac{1}{2}} .$$

It results from the inequalities of the theorem 5.2 that

$$\| r \|_{H_1} \leq C_\sigma C_\lambda e^{2\lambda_M} |\tau| \varepsilon \{(\varepsilon + \varepsilon_1)(\varepsilon + \varepsilon_1 e^{\lambda_M} |\tau|)\}^{\frac{1}{2}}$$

We recall that for the transverse part $h_{TT} = q$ it holds that

$$\| Dq \| = \| q \| \leq \| h \| + \| r \|$$

therefore

$$\| Dh \| \leq e^{\lambda_M} \varepsilon \{\sqrt{2} + C_\sigma e^{\lambda_M} |\tau| (\varepsilon + \varepsilon_1 e^{\lambda_M} |\tau|)^{\frac{1}{2}} (\varepsilon + \varepsilon_1)^{\frac{1}{2}}\}.$$

Definition 5.2 *We say that the hypothesis H_E is satisfied if there exists a positive number c_E such that $\varepsilon + \varepsilon_1 \leq c_E$. We denote by C_E any continuous and positive function of $c_E \in R^+$.*

If the hypothesis H_σ, H_λ and H_E are satisfied, h verifies the inequality:

$$\| Dh \| \leq C_\lambda |\tau|^{-1} \varepsilon (1 + C_\sigma C_\lambda C_E).$$

5.4 Estimates for the conformal factor λ.

The conformal factor λ satisfies on each Σ_t the equation

$$\Delta\lambda = f(\lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3$$

where the coefficients p_i are given by

$$p_1 = \frac{1}{4}\tau^2, p_2 = \frac{1}{2}(| h |^2 + | \dot{u} |^2), p_3 = -\frac{1}{2}(1 + | Du |^2)$$

The equation admits the subsolution λ_- given by

$$e^{-2\lambda_-} = \frac{1}{2}\tau^2$$

and it holds that

$$\lambda_- \leq \lambda_m \leq \lambda \leq \lambda_M \leq \lambda_+$$

with λ_+ a supersolution, for example:

$$\lambda_+ = \theta + v - minv$$

where v is the solution with mean value zero on Σ_t of the linear equation

$$\Delta v = f(\theta) \equiv p_1 e^{2\theta} - p_2 e^{-2\theta} + p_3$$

with[8] $e^{2\theta}$ a $t-$ dependent number, positive solution of the equation

$$\bar{p}_1 e^{4\theta} + \bar{p}_3 e^{2\theta} - \bar{p}_2 = 0,$$

where $\bar{f}$ denotes the mean value on (Σ, σ) of a function f :

$$\bar{f} \equiv \frac{1}{V_\sigma}\int_\Sigma f\mu_\sigma, \quad V_\sigma \equiv \int_\Sigma \mu_\sigma = -4\pi\chi.$$

Using the expressions of $\bar{p}_2, \bar{p}_3$ and $\bar{p}_1 = \frac{1}{4}\tau^2$, together with

$$\| \dot{u} \|^2 \leq e^{2\lambda_M} \| u' \|_g^2, \text{ and } \| h \|^2 \leq e^{2\lambda_M} \| h \|_g^2$$

and the expression of $\varepsilon^2 \equiv E(t)$ we have found in [CB-M1], section 8.1 (recall that $V_\sigma = -4\pi\chi$, a constant) that (we have renamed θ the $t-$ dependent number ω of CB-M1):

$$0 \leq \frac{1}{2}\tau^2 e^{2\theta} - 1 \leq V_\sigma^{-1}(1 + \frac{\tau^2}{2}e^{2\lambda_M})\varepsilon^2 \leq C_\lambda \varepsilon^2 \tag{5.11}$$

Lemma 5.3 *The following inequalities hold*

1.

$$0 \leq \lambda_M - \theta \leq 2 \| v \|_{L^\infty} . \tag{5.12}$$

2.

$$1 \leq \frac{1}{2}\tau^2 e^{2\lambda_M} \leq 1 + C_\lambda \varepsilon^2 + C_\lambda C_E \| v \|_{L^\infty} e^{4\|v\|_{L^\infty}} \tag{5.13}$$

[8]We have renamed θ the function called ω in [CB-M1].

PROOF. 1. It holds that

$$\lambda_M \leq \sup \lambda_+ = \theta + maxv - minv, \tag{5.14}$$

from which results 5.12.

2. The inequality 5.12 implies by elementary calculus

$$e^{2(\lambda_M - \theta)} \leq 1 + 4||v||_{L^\infty} e^{4||v||_{L^\infty}} \tag{5.15}$$

.The inequalities 5.11 and 5.15 imply that

$$\frac{1}{2}\tau^2 e^{2\lambda_M} \leq (1 + C_\lambda \varepsilon^2)(1 + 4 \parallel v \parallel_{L^\infty} e^{4\|v\|_{L^\infty}}), \tag{5.16}$$

from which the inequality 5.13 follows. ■

5.4.1 Estimate of v.

The equation satisfied by v implies

$$\int_\Sigma |Dv|^2 \mu_\sigma = -\int_\Sigma f(\theta) v \mu_\sigma$$

hence

$$\parallel Dv \parallel^2 \leq \parallel f(\theta) \parallel \parallel v \parallel$$

but the Poincaré inequality applied to the function v which has mean value 0 on Σ gives

$$\parallel v \parallel^2 \leq [\Lambda_\sigma]^{-1} \parallel Dv \parallel^2$$

where Λ_σ is the first (positive) eigenvalue of - Δ_σ for functions on Σ_t with mean value zero. Therefore on each Σ_t

$$\parallel Dv \parallel \leq [\Lambda_\sigma]^{-1/2} \parallel f(\theta) \parallel$$

We use Ricci identity and $R(\sigma) = -1$ to obtain that

$$\parallel \Delta_\sigma v \parallel^2 = \parallel D^2 v \parallel^2 - \frac{1}{2} \parallel Dv \parallel^2$$

The equation satisfied by v implies then, as in [1],

$$\|D^2 v\|^2 = \|f(\theta)\|^2 + \frac{1}{2}\|Dv\|^2$$

Assembling these various inequalities gives that:

$$\| v \|_{H_2} \leq [1 + 3/(2\Lambda_\sigma) + 1/\Lambda_\sigma^2]^{1/2} \| f(\theta) \|$$

The Sobolev inequality

$$\| v \|_{L^\infty} \leq C_\sigma \| v \|_{H_2}$$

gives then a bound on the L^∞ norm of v on Σ_t in terms of the L^2 norm of $f(\theta)$, a Sobolev constant C_σ and the lowest eigenvalue Λ_σ of $-\Delta_\sigma$, which is itself a number C_σ.

We now estimate the L^2 norm of $f(\theta)$.

$$f(\theta) \equiv f \equiv p_1 e^{2\theta} - p_2 e^{-2\theta} + p_3.$$

By the isoperimetric inequality, and since $\bar{f} = 0$, there exists a constant C_σ such that:

$$\|f\| \leq C_\sigma \|Df\|_1$$

We want to bound the right hand side in terms of the first and second energies of the wave map. We have by the definition of f and the expression of the $p's$ that:

$$\|Df\|_1 \leq \frac{1}{2}\{\|D|Du|^2\|_1 + e^{-2\theta}(\|D|h|^2\|_1 + \|D|\dot{u}|^2\|_1)\}$$

Lemma 5.4 *The following estimate holds under the hypothesis H_λ:*

$$\frac{1}{2} \| D|Du|^2 \|_1 \leq C_\lambda(\varepsilon^2 + \varepsilon\varepsilon_1)$$

Proof. We have:

$$D|Du|^2 = 2Du.\hat{D}^2 u$$

hence

$$\|D|Du|^2\|_1 \leq 2\|Du\|\|\hat{D}^2 u\|$$

We have seen that

$$\|Du\| = \| Du \|_g , \quad \|\hat{D}^2 u\| \leq e^{\lambda_M} \|\hat{\Delta}_g u\|_g + (1/\sqrt{2})\|Du\|_g \tag{5.17}$$

which implies the given result under the hypothesis H_λ and the definitions of ε and ε_1. ■

Lemma 5.5 *The following estimates hold under the hypothesis H_σ, H_E and H_λ :*

1.

$$\frac{1}{2}e^{-2\theta} \parallel D|h|^2 \parallel_1 \leq C_\sigma C_E C_\lambda \varepsilon^2$$

2.

$$\frac{1}{2}e^{-2\theta}\|D|\dot{u}|^2\|_1 \leq C_E C_\lambda C_\sigma(\varepsilon^2 + \varepsilon\varepsilon_1). \tag{5.18}$$

Proof. 1. We have:

$$\parallel D|h^2| \parallel_1 \leq 2 \parallel h \parallel \parallel Dh \parallel$$

Using the inequalities of sections 5.3.1 and 5.3.2 we find that

$$\parallel D|h^2| \parallel_1 \leq C_E C_\lambda \tau^{-2}\varepsilon^2$$

The given result follows from the bound 5.11 of $e^{-2\theta}$.

2. We recall the following estimates:

$$\parallel | u' |^2 \parallel \leq C_\lambda C_\sigma \tau^2\{\varepsilon^2 + \varepsilon\varepsilon_1\} \tag{5.19}$$

and

$$\parallel | Du |^2 \parallel \leq C_\lambda C_\sigma\{\varepsilon^2 + \varepsilon\varepsilon_1\} \tag{5.20}$$

and, by the Sobolev and Cauchy Schwarz inequalities

$$\parallel |h|^2 \parallel \leq C_\sigma\{\parallel h \parallel^2 + ||h|| \parallel Dh \parallel\}$$

which gives, using previous results

$$\parallel |h|^2 \parallel \leq C_\sigma C_\lambda C_E \tau^{-2}\varepsilon^2. \tag{5.21}$$

The estimate given in [CB-M1], lemma 21, when u is a scalar function holds when u is a wave map, , with the same proof which, as far as u is concerned, contains only norms. It gives the announced inequality. ■

We denote by $C_{E,\lambda,\sigma}$ a number depending only on c_E, c_λ and the considered compact domain of T_{eich}.

Lemma 5.6 *There exists a number $C_{E,\lambda,\sigma}$ such that the L^∞ norm of v is bounded by the following inequality*

$$\parallel v \parallel_\infty \leq C_{E,\lambda,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1) \tag{5.22}$$

Proof. Recall that there exists a Sobolev constant C_σ such that

$$\|v\|_\infty \leq C_\sigma\{\|D|Du|^2\|_1 + e^{-2\omega}(\|D|h|^2\|_1 + \|D|\dot{u}|^2\|_1)\}$$

The three terms in the sum have been evaluated in the lemma 5.4. ■

Theorem 5.7 *1.It holds that:*

$$1 \leq \frac{1}{2}\tau^2 e^{2\lambda_M} \leq 1 + C_{E,\lambda,\sigma}(\varepsilon + \varepsilon_1)^2, \tag{5.23}$$

2. There exists a number $\eta_1 > 0$ such that the hypothesis H_λ is satisfied, i.e.:

$$1 \leq \frac{1}{\sqrt{2}}|\tau|e^{\lambda_M} \leq c_\lambda, \quad c_\lambda > 1, \tag{5.24}$$

as soon as

$$\varepsilon + \varepsilon_1 \leq \eta_1. \tag{5.25}$$

Proof. 1. The lemmas 5.3 and 5.6.

2. By 5.23 it holds that, with $C_{E,\lambda,\sigma}$ the number of that inequality,

$$1 \leq \frac{1}{\sqrt{2}}|\tau|e^{\lambda_M} < c_\lambda \quad \text{if} \quad (\varepsilon + \varepsilon_1)^2 < \frac{c_\lambda^2 - 1}{C_{E,\lambda,\sigma}}. \tag{5.26}$$

The result follows from a continuity argument. ■

5.4.2 Bound of λ in H_1.

The following theorem holds, with the same proof as the theorem 23 of [CB-M1].

Theorem 5.8 *The following estimate holds*

$$||D\lambda||_{H_1} \leq C_{E,\lambda,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1). \tag{5.27}$$

6 Estimates in W_s^p.

6.1 Estimates for h in W_2^p.

The estimates of h in W_2^p, with $1 < p < 2$ (for definiteness we will choose $p = \frac{4}{3}$) will be obtained using estimates for the conformal factor λ which have been obtained by using the H_1 norm of h.

Theorem 6.1 *Under the H hypotheses there exists a positive number $C_{E,\lambda,\sigma}$ such that the W_2^p norm of h, choosing to be specific $p = \frac{4}{3}$, is bounded by*

$$\parallel h \parallel_{W_2^p} \leq C_{E,\lambda,\sigma} |\tau|^{-1} (\varepsilon + \varepsilon_1)$$

Corollary 6.2 *It holds that*

$$|\tau| \parallel h \parallel_\infty \leq C_{E,\lambda,\sigma} (\varepsilon + \varepsilon_1)$$

and that

$$\parallel h \parallel_{L^\infty(g)} \leq C_{E,\lambda,\sigma} |\tau| (\varepsilon + \varepsilon_1).$$

PROOF. The inequalities satisfied by $||h||_{W_2^p}$ in [CB-M1], with $p = \frac{4}{3}$, are still valid when u is a wave map, with the same proof, in particular because $\parallel Du.\dot{u} \parallel_{\frac{4}{3}}$ and $\parallel D(Du.\dot{u}) \parallel_{\frac{4}{3}}$ satisfy estimates of the same type as in [CB-M1]; indeed, using section 5.2:

$$\parallel Du.\dot{u} \parallel_{\frac{4}{3}} \leq \parallel Du \parallel \parallel \dot{u} \parallel_4 \leq C_{E,\lambda,\sigma} |\tau|^{-1} \varepsilon^{\frac{3}{2}} (\varepsilon + \varepsilon_1)^{\frac{1}{2}}.$$

On the other hand a straightforward calculation gives

$$D(Du.\dot{u}) \equiv \hat{D}(Du).\dot{u} + Du.\hat{D}\dot{u} \tag{6.1}$$

hence

$$\parallel D(Du.\dot{u}) \parallel_{\frac{4}{3}} \leq \parallel \hat{D}^2 u \parallel \parallel \dot{u} \parallel_4 + \parallel |Du| \parallel_4 \parallel \hat{D}\dot{u} \parallel$$

which gives, using previous estimates

$$\parallel D(Du.\dot{u}) \parallel_{\frac{4}{3}} \leq C_\lambda C_\sigma e^{\lambda_M} \{ \varepsilon^{\frac{1}{2}} (\varepsilon + \varepsilon_1)^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}} (\varepsilon + \varepsilon_1)^{\frac{1}{2}} \}$$

The result of the theorem follows from the bound of ε by $\varepsilon + \varepsilon_1$.

The corollary is a consequence of the Sobolev embedding theorem,

$$\parallel h \parallel_\infty \leq C_\sigma \parallel h \parallel_{W_2^p} \quad \text{if} \quad p > 1,$$

and the estimate .

$$\parallel h \parallel_{L^\infty(g)} = Sup_\Sigma \{ g^{ac} g^{bd} h_{ab} h_{cd} \}^{\frac{1}{2}} \leq e^{-2\lambda_m} \parallel h \parallel_\infty \leq \frac{1}{2} \tau^2 \parallel h \parallel_\infty$$

■

6.2 $\mathbf{W}_3^p$ estimates for N.

6.2.1 H_2 estimates of N.

Theorem 6.3 *There exists a number $C_{E,\lambda,\sigma}$ such that the H_2norm of N satisfies the inequality*

$$\| 2 - N \|_{H_2} \leq C_{E,\lambda,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1)$$

Corollary 6.4 *a. It holds that:*

$$\| 2 - N \|_{L^\infty} \leq C_{E,\lambda,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1) \tag{6.2}$$

b. There exists $\eta_2 > 0$ such that

$$\varepsilon + \varepsilon_1 \leq \eta_2 \tag{6.3}$$

implies the existence of a positive number N_m (independent of t) such that

$$N \geq N_m > 0.$$

PROOF. We write, as in [CB-M1] the equation satisfied by N in the form

$$\Delta(2 - N) - (2 - N) = \beta$$

with, having chosen the parameter t such that $\partial_t \tau = \tau^2$,

$$\beta \equiv (2 - N)(e^{2\lambda}\frac{1}{2}\tau^2 - 1) - N(e^{2\lambda} \mid u' \mid^2 + e^{-2\lambda} \mid h \mid^2)$$

The standard elliptic estimate applied to the form given to the lapse equation gives

$$\| 2 - N \|_{H_2} \leq C_\sigma \| \beta \|$$

Since $0 < N \leq 2$ and $e^{-2\lambda} \leq \frac{1}{2}\tau^2$ it holds that

$$\| \beta \| \leq 2(\frac{1}{2}e^{2\lambda_M}\tau^2 - 1)V_\sigma^{1/2} + 2(e^{2\lambda_M} \| |u'|^2 \| + \frac{1}{2}\tau^2 \| |h|^2 \|) \tag{6.4}$$

The L^4 norms of h and u' as well as $\frac{1}{2}e^{2\lambda_M}\tau^2 - 1$ have been estimated in the section on the conformal factor estimate. We deduce from these estimates the bound

$$\| \beta \| \leq C_{E,\lambda,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1).$$

which gives the result of the theorem.

The corollary a. is a consequence of the Sobolev embedding theorem, b. is a consequence of a. ∎

6.2.2 L^∞ estimate of DN.

Theorem 6.5 *Under the hypotheses H there exist numbers C_E, C_λ and C_σ such that if $1 < p < 2$, for instance $p = \frac{4}{3}$*

$$\| 2 - N \|_{W_3^p} \leq C_\lambda C_\sigma C_E (\varepsilon^2 + \varepsilon\varepsilon_1). \tag{6.5}$$

Corollary 6.6 *The gradient of N satisfies the inequality:*

$$\| DN \|_{L^\infty(g)} | \leq C_\lambda C_\sigma C_E |\tau| (\varepsilon^2 + \varepsilon\varepsilon_1) \tag{6.6}$$

Proof. The proof, essentially the same as in [CB-M1] rests on the W_1^p estimate of β, since applying the standard elliptic estimate gives

$$\| 2 - N \|_{W_3^p} \leq C_\sigma \| \beta \|_{W_1^p} \tag{6.7}$$

The estimate of

$$\| \beta \|_p \leq V_\sigma^{\frac{1}{p}-\frac{1}{2}} \| \beta \|$$

is the same as in [1] theorem 28. In the estimate $\| D\beta \|_p$the difference could be only in the estimate of the term $\| D|u'|^2 \|_p$. We have here

$$D|u'|^2 \equiv 2u'.\hat{D}u' \tag{6.8}$$

hence, with $p = \frac{4}{3}$

$$\| D|u'|^2 \|_p \leq 2 \| u' \|_4 \| Du' \| + 2e^{4(\gamma_M - \gamma_m)} ||D\gamma||_4 ||e^{-2\gamma}\omega'||_4^2$$

which leads to the same estimate as in [CB-M1]:

$$||\beta||_{W_1^p} \leq C_{E,\lambda,\sigma}(\varepsilon^2 + \varepsilon\varepsilon_1).$$

The corollary is a consequence of the Sobolev embedding theorem and the relation between σ and g norms:

$$\| DN \|_{L^\infty(g)} \leq e^{-\lambda_m} \| DN \|_\infty \leq e^{-\lambda_m} C_\sigma \| DN \|_{W_2^p} \leq C_{E,\lambda,\sigma} |\tau| (\varepsilon^2 + \varepsilon\varepsilon_1)$$

■

7 $\partial_t\sigma$ and shift estimates.

7.1 $\partial_t\sigma$ estimate.

We have chosen a section ψ of M_{-1} over Teichmuller space, denoted $\sigma \equiv \psi(Q)$ and supposed (Hypothesis H_σ) that Q remains in a compact subset of T_{eich}.

We then have:

$$\partial_t\sigma_{ab} = \frac{\partial\psi_{ab}}{\partial Q^I}\frac{dQ^I}{dt} \tag{7.1}$$

where $\frac{\partial\psi}{\partial Q^I}$ is uniformly bounded. Hence it holds that

$$|\partial_t\sigma| \leq C_\sigma|\frac{dQ}{dt}| \tag{7.2}$$

We recall that Q satisfies the differential equation

$$X_{IJ}\frac{dQ^I}{dt} + Y_{IJ}P^I + Z_J = 0$$

where $X_{IJ} \equiv \int_{\Sigma_t} X_I^{ab}X_{J,ab}\mu_{\sigma_t}$ is a matrix X with uniformly bounded inverse while Y_{IJ} and Z_J admit the following bounds, deduced from the basic estimates of N, λ and the L^2 bound of r :

$$|Y_{IJ}| \equiv |\int_{\Sigma_t} 2Ne^{-2\lambda}X_I^{ab}X_{J,ab}\mu_{\sigma_t}| \leq C_\sigma\tau^2$$

$$|Z_J| \equiv |\int_{\Sigma_t} 2Ne^{-2\lambda}r_{ab}X_J^{ab}\mu_{\sigma_t}| \leq C_\sigma\tau^2||r|| \leq C_\sigma C_E|\tau|\varepsilon(\varepsilon+\varepsilon_1).$$

On the other hand we recall that

$$q_{ab} \equiv h_{ab}^{TT} \equiv X_{I,ab}P^I \tag{7.3}$$

hence

$$P^I \equiv (X^{-1})^{IJ}\int_{\Sigma_t} X_J^{ab}q_{ab}\mu_\sigma. \tag{7.4}$$

Therefore:

$$|P^I| \leq C_\sigma||q|| \leq C_\sigma(||h|| + ||r||) \leq C_\sigma C_E|\tau|^{-1}(\varepsilon+\varepsilon_1) \tag{7.5}$$

hence

$$|Y_{IJ}P^J| \leq C_\sigma C_E|\tau|(\varepsilon+\varepsilon_1) \tag{7.6}$$

We have obtained inequalities of the following type:

$$|\frac{dQ}{dt}| \leq C_\sigma C_E |\tau|(\varepsilon+\varepsilon_1), \quad |\partial_t \sigma| \leq C_\sigma C_E |\tau|(\varepsilon+\varepsilon_1). \tag{7.7}$$

the derivatives $D^k \partial_t \sigma$ satisfy inequalities of the same type.

7.2 Shift estimate.

The equation to be satisfied by the shift ν reads, with n_a the covariant components of the vector ν in the metric σ, i.e. $n_a \equiv \sigma_{ab}\nu^b \equiv e^{-2\lambda} g_{ab}\nu^b$:

$$(L_\sigma n)_{ab} \equiv D_a n_b + D_b n_a - \sigma_{ab} D_c n^c = f_{ab} \tag{7.8}$$

$$f_{ab} \equiv 2N e^{-2\lambda} h_{ab} + \partial_t \sigma_{ab} - \frac{1}{2}\sigma_{ab}\sigma^{cd}\partial_t \sigma_{cd}$$

The elliptic theory for this first order system gives the estimate

$$||n||_{W_3^p} \equiv ||\nu||_{W_3^p} \leq C_\sigma ||f||_{W_2^p}. \tag{7.9}$$

with, if $p > 1$, using the bound 5.3 of $e^{-2\lambda}$,

$$||f||_{W_2^p} \leq C_\sigma C_E \{\tau^2 ||N||_{W_2^p} ||\lambda||_{W_2^p} ||h||_{W_2^p} + ||\partial_t \sigma||_{W_2^p}\}. \tag{7.10}$$

Hence, using previous estimates

$$||f||_{W_2^p} \leq C_\sigma C_E |\tau|(\varepsilon+\varepsilon_1). \tag{7.11}$$

8 Second energy estimate.

We have defined the energy $E^{(1)}(t)$ of gradient u by the formula

$$\tau^2 \varepsilon_1^2 \equiv E^{(1)}(t) \equiv \int_{\Sigma_t} (J_0 + J_1)\mu_g \tag{8.1}$$

with

$$J_1 = \frac{1}{2} \mid \hat{\Delta}_g u \mid^2 \equiv \frac{1}{2}\{2(\hat{\Delta}_g \gamma)^2 + \frac{1}{2} e^{-4\gamma}(\hat{\Delta}_g \omega)^2\} \tag{8.2}$$

$$J_0 = \frac{1}{2} \mid \hat{D} u' \mid_g^2 \equiv \frac{1}{2}\{2|\hat{D}\gamma'|_g^2 + \frac{1}{2} e^{-4\gamma}|\hat{D}\omega'|_g^2\} \tag{8.3}$$

8.1 Second energy equality.

We have:

$$\frac{d}{dt}\int_{\Sigma_t}(J_1+J_0)\mu_g=\int_{\Sigma_t}\{\partial_t(J_1+J_0)-(N\tau-\nabla_a\nu^a)(J_1+J_0)\}\mu_g \qquad (8.4)$$

On a compact manifold Σ, divergences integrate to zero, which leads to the following formula where the shift does not appear explicitly:

$$\frac{d}{dt}\int_{\Sigma_t}(J_1+J_0)\mu_g=\int_{\Sigma_t}\{\partial_0(J_1+J_0)-N\tau(J_1+J_0)\}\mu_g \qquad (8.5)$$

with, since $\hat{\partial}_0 G_{AB}=0$,

$$\partial_0 J_1=\hat{\partial}_0\hat{\Delta}_g u.\hat{\Delta}_g u \qquad (8.6)$$

We deduce from the commutation relation of the lemma 4.1 that

$$\hat{\partial}_0\hat{\Delta}_g u^A=g^{ab}(\hat{\nabla}_a\hat{\partial}_0\partial_b u^A-\partial_c u^A\hat{\partial}_0\Gamma^c_{ab})+\hat{\partial}_0 g^{ab}\hat{\nabla}_a\partial_b u^A+\hat{F}_1^A \qquad (8.7)$$

with

$$\hat{F}_1^A\equiv g^{ab}R_{CB}{}^A{}_D\partial_0 u^C\partial_a u^B\partial_b u^D. \qquad (8.8)$$

Hence, using the identities 4.23:

$$\hat{\partial}_0\hat{\Delta}_g u^A=g^{ab}\hat{\nabla}_a\hat{\partial}_0\partial_b u^A+N\tau\hat{\Delta}_g u^A+F_1^A+\hat{F}_1^A, \qquad (8.9)$$

with

$$F_1^A\equiv 2\partial_c u^A(h_g^{ac}\partial_a N+N\nabla_a k^{ac})+2Nh_g^{ab}\hat{\nabla}_a\partial_b u^A. \qquad (8.10)$$

We have therefore, using Stokes formula, and $\hat{\partial}_0\partial_b u\equiv\hat{\nabla}_b(Nu')$

$$\int_{\Sigma_t}\partial_0 J_1\mu_g=\int_{\Sigma_t}\{-N\hat{\nabla}_a u'.\hat{\nabla}^a\hat{\Delta}_g u+2N\tau J_1-\partial_a Nu'.\hat{\nabla}^a\hat{\Delta}_g u+(F_1+\hat{F}_1).\hat{\Delta}_g u\}\mu_g \qquad (8.11)$$

On the other hand

$$\partial_0 J_{0.}=g^{ab}\hat{\partial}_0\hat{\nabla}_a u'.\hat{\nabla}_b u'+N(h_g^{ab}+\frac{1}{2}g^{ab}\tau)\partial_a u'.\partial_b u'$$

where we have used the identity, h_g^{ab} denoting the contravariant components of h_{ab} computed with the metric g,

$$\hat{\partial}_0 g^{ab}=2Nk^{ab}\equiv 2Nh_g^{ab}+Ng^{ab}\tau.$$

The commutation relation 4.20 gives that:

$$g^{ab}\hat{\partial}_0\hat{\nabla}_a u'.\hat{\nabla}_b u' \equiv \hat{\partial}_0\hat{\nabla}_a u'.\hat{\nabla}^a u' = \hat{\nabla}_a\hat{\partial}_0 u'.\hat{\nabla}^a u' + \hat{F}_0, \tag{8.12}$$

with

$$\hat{F}_0 \equiv R_{AB,CD} u'^D \partial_0 u^A \partial_a u^B \hat{\nabla}^a u'^C \tag{8.13}$$

Therefore, using the wave map equation

$$-\hat{\partial}_0 u' + N\hat{\Delta}_g + \partial^a N \partial_a u + N\tau u' = 0. \tag{8.14}$$

we find that

$$\hat{\partial}_0\hat{\nabla}_a u'.\hat{\nabla}^a u' = \hat{\nabla}_a[N\hat{\Delta}_g u + \partial^c N\partial_c u) + N\tau u'].\hat{\nabla}^a u' + \hat{F}_0 \tag{8.15}$$

Therefore

$$\int_{\Sigma_t} \partial_0 J_{0.}\mu_g = \int_{\Sigma_t} \{N\hat{\nabla}^a\hat{\Delta}_g u.\hat{\nabla}^a u' + 3N\tau J_0 + F_0 + \hat{F}_0\}\mu_g \tag{8.16}$$

with

$$F_0 \equiv [\partial^a N\hat{\Delta}_g u + \hat{\nabla}^a(\partial^c N\partial_c u)].\hat{\nabla}_a u' + \tau\partial^a N u'.\hat{\nabla}_a u' + +Nh_g^{ab}\hat{\nabla}_a u'.\hat{\nabla}_b u' \tag{8.17}$$

We see that the third order terms in u disappear from the integral of $\partial_0(J_{0.} + J_1)$ which reduces to

$$\int_{\Sigma_t} \partial_0(J_{0.} + J_1)\mu_g = \int_{\Sigma_t} \{3N\tau J_0 + 2N\tau J_1\}\mu_g + Z_1 \tag{8.18}$$

with

$$Z_1 \equiv \int_{\Sigma_t} \{(F_1 + \hat{F}_1).\hat{\Delta}_g u + (F_0 + \hat{F}_0\}\mu_g \tag{8.19}$$

We have obtained

$$\frac{dE^{(1)}}{dt} = \int_{\Sigma_t} N\tau(2J_0 + J_1)\mu_g + Z_1 \tag{8.20}$$

which we write

$$\frac{dE^{(1)}}{dt} - 2\tau E^{(1)} = \tau\int_{\Sigma_t} NJ_0\mu_g + Z_2 + Z_1 \tag{8.21}$$

with

$$Z_2 \equiv \tau\int_{\Sigma_t} (N-2)(2J_0 + J_1)\mu_g. \tag{8.22}$$

8.2 Second energy inequality.

Since τ is negative (and N positive) the equality 8.21 implies the inequality

$$\frac{dE^{(1)}}{dt} - 2\tau E^{(1)}(t) \leq Z_1 + Z_2. \tag{8.23}$$

We now estimate the various terms of Z_1, Z_2, called non linear terms because they are all homogeneous and cubic in h_g^{ab}, $N-2$, and the derivatives of N and u. These estimates are essentially the same as the ones given in [CB-M1], due to the estimates of the previous section. We first write, using the estimate of $||N-2||_{L^\infty(g)}$ and the definition of ε_1 :

$$|Z_2| \equiv |\tau \int_{\Sigma_t} (N-2)(2J_0 + J_1)\mu_g| \leq C_\lambda C_\sigma C_E |\tau|^3(\varepsilon + \varepsilon_1)^4. \tag{8.24}$$

We now estimate the different terms of Z_1, beginning with the terms X_1 and X_2 coming from F_0 :

$$|X_1| \equiv | \int_{\Sigma_t} \{(\partial^a N \hat{\Delta}_g u + \partial^c N \hat{\nabla}^a \partial_c u).\hat{\nabla}_b u'\}\mu_g|$$

A proof analogous to the proof of the lemma 4.2 gives

$$|\hat{\nabla} Du|_g^2 \leq |\hat{\Delta}_g u|_g^2 - \frac{1}{2}R(g)|Du|_g^2. \tag{8.25}$$

The hamiltonian constraint 2.21 implies that

$$R(g) = |u'|^2 + |Du|_g^2 + |h|_g^2 - \frac{1}{2}\tau^2 \geq -\frac{1}{2}\tau^2 \tag{8.26}$$

therefore

$$||\hat{\nabla} Du||_g^2 \leq ||\hat{\Delta}_g u||_g^2 + \frac{1}{4}\tau^2 ||Du||_g^2 \tag{8.27}$$

Using the estimate 6.6 of DN gives then

$$|X_1| \leq C_{E,\lambda,\sigma}|\tau|^3(\varepsilon + \varepsilon_1)^4 \tag{8.28}$$

The remaining, X_2, of the integral of F_0 is estimated as follows

$$|X_2| = | \int_{\Sigma_t} (\hat{\nabla}_a \partial^c N)\partial_c u)].\hat{\nabla}_b u' \mu_g| \leq (||\nabla^2 N||_{L^4(g)}||\ |Du|\ ||_{L^4(g)}||\hat{\nabla} u'||_g \tag{8.29}$$

The estimates of $||\nabla^2 N||_{L^4(g)}$ and $||\ |Du|\ ||_{L^4(g)}$ given in [CB-M1], section 10.2.2, for the estimate of Y_2 applies here, due to the lemma 5.1, and give

$$|X_2| \leq C_{E,\lambda,\sigma}|\tau|^3(\varepsilon+\varepsilon_1)^4. \tag{8.30}$$

The terms in

$$X_3 \equiv \int_{\Sigma_t} F_1.\hat{\Delta}_g u\mu_g \equiv \int_{\Sigma_t} [2\partial_c u(h_g^{ac}\partial_a N + N\nabla_a k^{ac}) + 2Nh_g^{ab}\hat{\nabla}_a\partial_b u].\hat{\Delta}_g u\mu_g \tag{8.31}$$

are analogous to terms found in [CB-M1] and can be estimated similarly, giving an inequality of the form

$$|X_3| \leq C_{E,\lambda,\sigma}|\tau|^3(\varepsilon+\varepsilon_1)^3. \tag{8.32}$$

The new terms, in our unpolarized case, are

$$X_4 \equiv \int_{\Sigma_t} \hat{F}_1.\hat{\Delta}_g u\mu_g \equiv \int_{\Sigma_t} g^{ab}R_{CB,AD}\partial_0 u^C\partial_a u^B\partial_b u^D\hat{\Delta}_g u^A\mu_g \tag{8.33}$$

and

$$X_5 \equiv \int_{\Sigma_t} \hat{F}_0\mu_g \equiv \int_{\Sigma_t} R_{AB,CD}u'^D\partial_0 u^A\partial^a u^B\hat{\nabla}_a u'^C\mu_g \tag{8.34}$$

We have here $|Riemann(G)| = 4$, therefore, using $\partial_0 u \equiv Nu'$ and $0 < N \leq 2$, and the Hölder inequality:

$$|X_4| \leq 8\int_{\Sigma_t} |u'|\ |Du|_g^2|\hat{\Delta}_g u|_g\mu_g \leq 8|\tau|\varepsilon_1||u'||_{L^6(g)}||\ |Du|^2\ ||_{L^3(g)}. \tag{8.35}$$

and

$$|X_5| \leq 8||u'||^2_{L^6(g)}||\ |Du|_g\ ||_{L^6(g)}|\tau|\varepsilon_1 \tag{8.36}$$

The $L^6(g)$ norms can be estimated as follows. It results from the definitions that:

$$||\ |Du|_g\ ||_{L^6(g)} = \{\int_{\Sigma_t} e^{-4\lambda}|Du|^6\mu_\sigma\}^{\frac{1}{6}} \leq e^{-\frac{2}{3}\lambda_m}||\ |Du|\ ||_6 \tag{8.37}$$

while, by the Sobolev embedding theorem

$$||\ |Du|\ ||_6 \leq C_\sigma(||Du|| + ||\ D|Du|\ || \tag{8.38}$$

It holds that

$$D|Du| = \frac{D|Du|^2}{2|Du|} = \frac{\hat{D}Du.Du}{|Du|} \tag{8.39}$$

hence

$$|D|Du|\;| \leq |\hat{D}Du| \tag{8.40}$$

and

$$||\;|Du|\;||_6 \leq C_\sigma(||Du|| + ||\;\hat{D}Du\;||) \tag{8.41}$$

The inequalities of the lemma 5.1 and the lower bound of λ_m give then

$$||\;|Du|\;||_{L^6(g)} \leq C_\sigma C_\lambda |\tau|^{\frac{2}{3}}(\varepsilon + \varepsilon_1)) \tag{8.42}$$

An analogous proof gives that

$$||u'||_{L^6(g)} \leq e^{\frac{1}{3}\lambda_M}||u'||_6 \tag{8.43}$$

and

$$||u'||_6 \leq C_\sigma(||u'|| + ||D|u'|\;||), \tag{8.44}$$

with

$$|D|u'|\;| \leq |\hat{D}u'|, \quad ||\hat{D}u'|| = ||\hat{D}u'||_g \tag{8.45}$$

therefore, using again the lemma 5.1

$$||u'||_{L^6(g)} \leq C_\sigma e^{\frac{1}{3}\lambda_M}|\tau|(\varepsilon + \varepsilon_1) \leq C_\sigma C_\lambda |\tau|^{\frac{2}{3}}(\varepsilon + \varepsilon_1). \tag{8.46}$$

These estimates imply that

$$|X_4| \leq C_\sigma C_\lambda |\tau|^3(\varepsilon + \varepsilon_1)^3 \varepsilon_1 \tag{8.47}$$

and the same inequality for X_5. We have proved the following theorem

Theorem 8.1 *The second energy satisfies an equality of the form*

$$\frac{dE^{(1)}}{dt} - 2\tau E^{(1)}(t) \leq |\tau|^3 B_1. \tag{8.48}$$

with

$$|B_1| \leq C_\sigma C_\lambda C_E(\varepsilon + \varepsilon_1)^3.$$

9 Corrected energies.

9.1 Corrected first energy

9.1.1 Definition and lower bound.

One defines as follows a corrected first energy where α is a constant, which we will choose positive:

$$E_\alpha(t) = E(t) - \alpha\tau E_c(t), \qquad E_c(t) \equiv \int_{\Sigma_t} (u - \bar{u}).u'\mu_g \tag{9.1}$$

where we have set:

$$(u - \bar{u}).u' \equiv 2(\gamma - \bar{\gamma})\gamma' + \frac{1}{2}e^{-4\gamma}(\omega - \bar{\omega})\omega' \tag{9.2}$$

and denoted by $\bar{f}$ the mean value on (Σ_t, σ) of a scalar function f :

$$\bar{f} = \frac{1}{Vol_\sigma(\Sigma_t)} \int_{\Sigma_t} f\mu_\sigma \equiv -\frac{1}{4\pi\chi} \int_{\Sigma_t} f\mu_\sigma$$

An estimate of E_α will involve second derivatives of u, it cannot alone gives a bound of the first energy E.

The Cauchy Schwarz inequality on (Σ, g) and the relation between g and σ imply that:

$$|\int_{\Sigma_t} (\gamma - \bar{\gamma})\gamma'\mu_g| \leq ||\gamma - \bar{\gamma}||_g \parallel \gamma' \parallel_g \leq e^{\lambda_M} ||\gamma - \bar{\gamma}|| \; ||\gamma'||_g \tag{9.3}$$

Using the Poincaré inequality and recalling that Λ_σ denotes the first positive eigen value of the laplacian Δ_σ on functions with mean value zero gives the majoration (recall that $\parallel Df \parallel = \parallel Df \parallel_g$ if f is a scalar function):

$$|\int_{\Sigma_t} (\gamma - \bar{\gamma})\gamma'\mu_g| \leq e^{\lambda_M} \Lambda_\sigma^{-1/2} \parallel D\gamma \parallel \parallel \gamma' \parallel_g \tag{9.4}$$

an analogous procedure gives, with γ_m and γ_M the lower and upper bounds of γ :

$$|\int_{\Sigma_t} e^{-4\gamma}(\omega - \bar{\omega})\omega'\mu_g| \leq e^{2(\gamma_M - \gamma_m)} e^{\lambda_M} \Lambda_\sigma^{-1/2} \parallel e^{-2\gamma} D\omega \parallel \parallel e^{-2\gamma}\omega' \parallel_g \tag{9.5}$$

since

$$\| D\omega \| \leq e^{2\gamma_M} \| e^{-2\gamma} D\omega \| .$$

Using the definition of the $G-$norm we see that the inequalities 9.4, 9.5 imply:

$$| \int_{\Sigma_t} (u - \bar{u}).u' \mu_g | \leq e^{\lambda_M} \Lambda_\sigma^{-\frac{1}{2}} e^{2(\gamma_M - \gamma_m)} \| Du \| \| u' \|_g \tag{9.6}$$

with (theorem 5.7)

$$e^{\lambda_M} \leq |\tau|^{-1} \{ \sqrt{2} + C_\sigma C_E C_\lambda (\varepsilon + \varepsilon_1) \}. \tag{9.7}$$

Lemma 9.1 *It holds that:*

$$\gamma_M - \gamma_m \leq C_\sigma C_E C_\lambda \{ \varepsilon + \varepsilon_1 \}.$$

PROOF. We have:

$$0 \leq \gamma_M - \gamma_m \leq 2 \| \gamma - \bar{\gamma} \|_{L^\infty}$$

The Sobolev imbedding theorem gives therefore that:

$$\gamma_M - \gamma_m \leq 2C_\sigma \| \gamma - \bar{\gamma} \|_{H_2}$$

hence, using again the Poincaré inequality to estimate $||\gamma - \bar{\gamma}||$,

$$\gamma_M - \gamma_m \leq 2C_\sigma \{ (\Lambda_\sigma^{-1} + 1) \| D\gamma \| + \| D^2\gamma \| \}.$$

It results from the definitions of the G norm and of ε that

$$||D\gamma||^2 \leq \frac{1}{2} ||Du||^2 \leq \varepsilon^2. \tag{9.8}$$

On the other hand since γ is a scalar function on a 2-manifold with constant scalar curvature -1, it holds that:

$$||D^2\gamma||^2 = ||\Delta\gamma||^2 + \frac{1}{2} ||D\gamma||^2 \tag{9.9}$$

We have

$$\Delta\gamma = \hat{\Delta}\gamma - \frac{1}{2} e^{-4\gamma} |D\omega|^2$$

hence

$$||\Delta\gamma|| \leq ||\hat{\Delta}\gamma|| + \frac{1}{2} || \, |e^{-2\gamma} D\omega|^2 || \tag{9.10}$$

with

$$||\hat{\Delta}\gamma||^2 \leq \frac{1}{2}||\hat{\Delta}u||^2 \leq \frac{1}{2}e^{2\lambda_M}||\hat{\Delta}_g u||^2 \leq C_\lambda \varepsilon_1^2, \tag{9.11}$$

and, using the lemma 5.1

$$||\ |e^{-2\gamma}D\omega|^2|| \leq ||\ |Du|^2|| \leq C_\sigma C_\lambda\{\varepsilon^2 + \varepsilon\varepsilon_1\}. \tag{9.12}$$

hence:

$$||D^2\gamma||^2 \leq C_E C_\lambda C_\sigma(\varepsilon_1^2 + \varepsilon^2) \tag{9.13}$$

Using the hypothesis H_E we deduce from all these inequalities the announced result. ■

We deduce from this lemma and the elementary calculus formula:

$$e^{2(\gamma_M - \gamma_m)} \leq 1 + 2(\gamma_M - \gamma_m)e^{2(\gamma_M - \gamma_m)}$$

that there exists an inequality of the form

$$e^{2(\gamma_M - \gamma_m)} \leq 1 + C_{E,\lambda,\sigma}(\varepsilon + \varepsilon_1). \tag{9.14}$$

We have proved that

$$|\tau \int_{\Sigma_t}(u - \bar{u}).u'\mu_g| \leq \sqrt{2}\Lambda_\sigma^{-\frac{1}{2}} \parallel Du \parallel\parallel u' \parallel_g + A_1 \tag{9.15}$$

with

$$|A_1| \leq C_{E,\lambda,\sigma}\varepsilon^2(\varepsilon + \varepsilon_1). \tag{9.16}$$

Therefore:

$$E_\alpha(t) \geq Q_{\alpha,\Lambda}(x_0, x_1) - C_{E,\lambda,\sigma}\varepsilon^2(\varepsilon + \varepsilon_1). \tag{9.17}$$

where $\parallel Du \parallel = x_1$, $\parallel u' \parallel_g = x_0$, $x = (x_0, x_1)$ and $Q_{\alpha,\Lambda}$ is the quadratic form

$$Q_{\alpha,\Lambda}(x) \equiv \frac{1}{2}(x_0^2 + x_1^2) - \alpha\sqrt{2}\Lambda_\sigma^{-\frac{1}{2}}x_0x_1. \tag{9.18}$$

The right hand side of the inequality 9.17 can be positive only if this quadratic form is positive, that is if:

$$\alpha < \frac{\Lambda_\sigma^{\frac{1}{2}}}{\sqrt{2}}. \tag{9.19}$$

There exists a number $K > 0$ such that:

$$Q_{\alpha,\Lambda}(x) \geq KE(t) \equiv \frac{1}{2}K(x_0^2 + x_1^2) \tag{9.20}$$

if the following quadratic form Q_K is positive definite

$$Q_K(x) \equiv (1 - K)(x_0^2 + x_1^2) - 2\alpha\sqrt{2}\Lambda_\sigma^{-\frac{1}{2}} x_0 x_1) \tag{9.21}$$

that is, under the condition 9.19 on α,

$$0 < K < 1 - \alpha\sqrt{2}\Lambda_\sigma^{-\frac{1}{2}}. \tag{9.22}$$

There will then exist a number $0 < \ell < K$ such that

$$E_\alpha(t) \geq \ell E(t) \tag{9.23}$$

as soon as

$$\varepsilon + \varepsilon_1 \leq \eta_3 \tag{9.24}$$

with ($C_{\sigma,E,\lambda}$ denotes the coefficient of this type in 9.17)

$$\eta_3 = \frac{K - \ell}{C_{\sigma,E,\lambda}} < \frac{1 - \alpha\sqrt{2}\Lambda_\sigma^{-\frac{1}{2}}}{C_{\sigma,E,\lambda}}. \tag{9.25}$$

9.2 Time derivative of the corrected first energy.

We have (recall that terms involving the shift are exact divergences which integrate to zero, and we have set $\frac{d\tau}{dt} = \tau^2$):

$$\frac{dE_\alpha}{dt} = \frac{dE}{dt} - \alpha\tau\mathcal{R} \quad \text{with} \quad \mathcal{R} \equiv \frac{dE_c}{dt} + \tau E_c,$$

that is:

$$\mathcal{R} \equiv \int_{\Sigma_t} \{\hat{\partial}_0 u'.(u - \bar{u}) + u'.\hat{\partial}_0(u - \bar{u}) - N\tau u'.(u - \bar{u}) + \tau u'.(u - \bar{u})\}\mu_g \tag{9.26}$$

The mapping u satisfies the wave map equation:

$$-\hat{\partial}_0 u' + \hat{\nabla}^a(N\partial_a u) + \tau N u' = 0, \tag{9.27}$$

therefore, performing an integration by parts where we derivate $u-\bar{u}$ as if it were a section of $E^{0,1}$ we obtain that:

$$\int_{\Sigma_t} (\hat{\partial}_0 u' - N\tau u').(u-\bar{u})\mu_g = \int_{\Sigma_t} \hat{\nabla}^a (N\partial_a u).(u-\bar{u})\mu_g \tag{9.28}$$

$$= \int_{\Sigma_t} -Ng^{ab}\partial_a u.\hat{\partial}_b(u-\bar{u})\mu_g \tag{9.29}$$

where

$$\hat{\partial}_b(u^B - \bar{u}^B) \equiv \partial_b(u^B - \bar{u}^B) + G^B_{CD}\partial_b u^C(u^D - \bar{u}^D) \tag{9.30}$$

that is, due to the values of the coefficients G^A_{BC} (remark 2.2):

$$\hat{\partial}_b(\gamma - \bar{\gamma}) \equiv \partial_b(\gamma - \bar{\gamma}) + \frac{1}{2}e^{-4\gamma}\partial_b\omega(\omega - \bar{\omega}), \tag{9.31}$$

$$\hat{\partial}_b(\omega - \bar{\omega}) \equiv \partial_b(\omega - \bar{\omega}) - 2e^{-4\gamma}[\partial_b\omega(\gamma - \bar{\gamma}) + \partial_b\gamma(\omega - \bar{\omega})] \tag{9.32}$$

hence, using the expression of the metric G :

$$\int_{\Sigma_t} -Ng^{ab}\partial_a u.\hat{\partial}_b(u-\bar{u})\mu_g = -\int_{\Sigma_t} N\{|Du|^2_g - e^{-4\gamma}g^{ab}\partial_a\omega\partial_b\omega(\gamma - \bar{\gamma})\}\mu_g \tag{9.33}$$

The non linear term can be estimated using previous results, namely (recall $g^{ab}\mu_g = \sigma^{ab}\mu_\sigma$):

$$|\int_{\Sigma_t} Ne^{-4\gamma}g^{ab}\partial_a\omega\partial_b\omega(\gamma-\bar{\gamma})\mu_g| \leq 2||\gamma - \bar{\gamma}||\ ||\ |e^{-2\gamma}D\omega|^2||. \tag{9.34}$$

It holds that

$$||\gamma - \bar{\gamma}|| \leq \Lambda_\sigma^{-\frac{1}{2}}||D\gamma||, \quad ||D\gamma||^2 \leq \varepsilon^2 \tag{9.35}$$

and, due to the definition of the norm in G and the lemma 5.1,

$$||\ |e^{-2\gamma}D\omega|^2|| \leq 2||\ |Du|^2|| \leq C_\sigma C_\lambda(\varepsilon + \varepsilon_1)^2 \tag{9.36}$$

On the other hand:

$$\hat{\partial}_0(u-\bar{u})^A = \partial_0(u-\bar{u})^A + G^A_{CD}\partial_0 u^C(u-\bar{u})^D. \tag{9.37}$$

A straightforward computation using the values of the coefficients G^A_{CD} gives that

$$\int_{\Sigma_t} u'.\hat{\partial}_0(u-\bar{u})\mu_g = \int_{\Sigma_t} \{N|u'|^2 - Ne^{-4\gamma}\omega'^2(\gamma - \bar{\gamma})\}\mu_g$$

$$-\partial_t\bar{\gamma}\int_{\Sigma_t}\frac{1}{2}\gamma'\mu_g - \partial_t\bar{\omega}\int_{\Sigma_t}2e^{-4\gamma}\omega'\mu_g. \tag{9.38}$$

The non linear term can be estimated as before:

$$\int_{\Sigma_t}Ne^{-4\gamma}\omega'^2(\gamma-\bar{\gamma})\mu_g \leq 2e^{2\lambda_M}\Lambda_\sigma^{-\frac{1}{2}}\varepsilon||\,|u'|^2|| \leq C_\sigma C_\lambda\varepsilon^2(\varepsilon+\varepsilon_1) \tag{9.39}$$

To bound the remaining terms we observe that for a scalar function f , since $V_\sigma = -4\pi\chi$ is a constant, it holds that

$$\partial_t\bar{f} = \frac{1}{V_\sigma}\partial_t\int_{\Sigma_t}f\mu_\sigma = \frac{1}{V_\sigma}\int_{\Sigma_t}\{\partial_0 f + \nu^a\partial_a f + \frac{1}{2}f\sigma^{ab}\partial_t\sigma_{ab}\}\mu_\sigma \tag{9.40}$$

with, for the considered metric σ

$$\int_{\Sigma_t}\sigma^{ab}\partial_t\sigma_{ab}\mu_\sigma = 0. \tag{9.41}$$

We write $\partial_t\bar{f}$ under the form (recall that $f' \equiv N^{-1}\partial_0 f$)

$$\partial_t\bar{f} = \frac{1}{V_\sigma}\int_{\Sigma_t}\{2f' + (N-2)f' + \nu^a\partial_a f + \frac{1}{2}(f-\bar{f})\sigma^{ab}\partial_t\sigma_{ab}\}\mu_\sigma \tag{9.42}$$

with

$$\frac{1}{V_\sigma}\int_{\Sigma_t}2f'\mu_\sigma \equiv \frac{\tau^2}{V_\sigma}\int_{\Sigma_t}f'\mu_g + \frac{1}{V_\sigma}\int_{\Sigma_t}(2-e^{2\lambda}\tau^2)f'\mu_\sigma \tag{9.43}$$

We deduce from these equalities that

$$-\partial_t\bar{\gamma}\int_{\Sigma_t}\gamma'\mu_g = -\frac{2}{V_\sigma}(\int_{\Sigma_t}\gamma'\mu_g)^2 + \frac{1}{V_\sigma}X \tag{9.44}$$

with:

$$X \equiv \{\int_{\Sigma_t}[(2-e^{2\lambda}\tau^2+N-2)\gamma' + \nu^a\partial_a\gamma + \frac{1}{2}(\gamma-\bar{\gamma})\sigma^{ab}\partial_t\sigma_{ab}]\mu_\sigma\}\{\int_{\Sigma_t}\gamma'\mu_g\} \tag{9.45}$$

The equality 9.44 implies the inequality:

$$-\partial_t\bar{\gamma}\int_{\Sigma_t}\gamma'\mu_g \leq \frac{1}{V_\sigma}X. \tag{9.46}$$

All the terms in X are non linear in the energies and can be estimated. Indeed:

$$|\int_{\Sigma_t}\gamma'\mu_g| \leq V_g^{\frac{1}{2}}||\gamma'||_g \leq e^{\lambda_M}V_\sigma||\gamma'||_g \tag{9.47}$$

while (theorems 5.7 and corollary 6.4)

$$|2-e^{2\lambda}\tau^2+N-2| \leq C_E C_\lambda C_\sigma(\varepsilon+\varepsilon_1)^2 \tag{9.48}$$

and

$$\int_{\Sigma_t}|\gamma'|\mu_\sigma \leq V_\sigma^{\frac{1}{2}}||\gamma'|| \leq V_\sigma^{\frac{1}{2}}e^{-\lambda_m}||\gamma'||_g, \tag{9.49}$$

Also

$$||\nu|| \leq C_E C_\sigma|\tau|(\varepsilon+\varepsilon_1) \tag{9.50}$$

and

$$||D\gamma|| = ||D\gamma||_g \leq \varepsilon \tag{9.51}$$

Using section 7.1 we find that:

$$|\sigma^{ab}\partial_t\sigma_{ab}| \leq C_\sigma C_E|\tau|(\varepsilon+\varepsilon_1). \tag{9.52}$$

The same type of inequalities applies to the scalar function $\partial_t\bar{\omega}$, but we must now use also the identities

$$e^{-2\bar{\gamma}}\int_{\Sigma_t}\omega'\mu_g \equiv \int_{\Sigma_t}e^{-2\gamma}\omega' e^{2(\gamma-\bar{\gamma})}\mu_g \equiv \int_{\Sigma_t}\{e^{-2\gamma}\omega' + (e^{2(\gamma-\bar{\gamma})}-1)e^{-2\gamma}\omega'\}\mu_g$$

$$e^{2\bar{\gamma}}\int_{\Sigma_t}e^{-4\gamma}\omega'\mu_g \equiv \int_{\Sigma_t}e^{-2\gamma}\omega' e^{2(\bar{\gamma}-\gamma)}\mu_g \equiv \int_{\Sigma_t}\{e^{-2\gamma}\omega' + (e^{-2(\gamma-\bar{\gamma})}-1)e^{-2\gamma}\omega'\}\mu_g$$

to obtain an inequality which bounds $-\partial_t\bar{\omega}\int_{\Sigma_t}e^{-4\gamma}\omega'\mu_g$ with higher order terms in the energies, using the bound

$$|e^{\pm 2(\gamma-\bar{\gamma})}-1| \leq 2|\gamma-\bar{\gamma}|e^{2|\gamma-\bar{\gamma}|} \leq C_{E,\lambda,\sigma}(\varepsilon+\varepsilon_1). \tag{9.53}$$

We have proved that

$$\mathcal{R} \leq \int_{\Sigma_t}\{-N|Du|^2 + N|u'|^2 + \tau u'.(u-\bar{u})\}\mu_g + A_2 \tag{9.54}$$

with:

$$|A_2| \leq C_{E,\lambda,\sigma}(\varepsilon+\varepsilon_1)^3. \tag{9.55}$$

Theorem 9.2 *There exist numbers $\alpha > 0$ and $k > 0$ such that*

$$\frac{dE_\alpha}{dt} - k\tau E_\alpha \leq |\tau A| \tag{9.56}$$

with

$$|\tau A| \leq |\tau| C_{E,\lambda,\sigma}(\varepsilon + \varepsilon_1)^3. \tag{9.57}$$

1. If $\Lambda > \frac{1}{8}$ the best choice is

$$\alpha = \frac{1}{4},\ k = 1 \tag{9.58}$$

2. $\Lambda \leq \frac{1}{8}$. Then α and k are such that:

$$0 < \alpha < \frac{4}{8 + \Lambda_\sigma^{-1}} \leq \frac{1}{4}, \quad 0 < k < 1 - \frac{1 - 4\alpha}{(1 - 2\Lambda_\sigma^{-1}\alpha^2)^{\frac{1}{2}}} \tag{9.59}$$

A number α satisfying the conditions of the above theorem is also such that $0 < \alpha < \frac{\Lambda_\sigma^{\frac{1}{2}}}{\sqrt{2}}$.

Proof. using 10.54 and the expression 4.8 of $\frac{dE}{dt}$ we find that:

$$\frac{dE_\alpha}{dt} \leq \tau \int_{\Sigma_t} \{|h|^2 + (1-2\alpha)|u'|^2 + 2\alpha|Du|_g^2] - \alpha\tau u'.(u - \bar{u})\}\mu_g + |\tau A| \tag{9.60}$$

where

$$A \equiv \alpha A_1 + A_2. \tag{9.61}$$

with

$$A_2 \equiv \int_{\Sigma_t} \frac{1}{2}(N-2)[|h|^2 + (1-2\alpha)|u'|^2 + 2\alpha|Du|_g^2]\mu_g \tag{9.62}$$

We deduce from the corollary 6.4 (L^∞ estimate of $N - 2$) that A_2 satisfies the same type of estimate than A_1, hence:

$$|\tau A| \leq |\tau| C_E C_\lambda C_\sigma (\varepsilon + \varepsilon_1)^3. \tag{9.63}$$

We look for a positive number k such that the difference $\frac{dE_\alpha}{dt} - k\tau E_\alpha$ can be estimated with higher order terms in the energies. We deduce from 9.60 that:

$$\frac{dE_\alpha}{dt} - k\tau E_\alpha \leq \tau\{||h||_g{}^2 + (1 - 2\alpha - \frac{k}{2})||u'||_g{}^2 + (2\alpha - \frac{k}{2})||Du||_g^2 \tag{9.64}$$

$$+\alpha\int_{\Sigma_t}|\tau|(1-k)u'.(u-\bar{u})\mu_g\}+|\tau A| \tag{9.65}$$

We have seen that:

$$|\tau\int_{\Sigma_t}u'.(u-\bar{u})\mu_g|\leq\sqrt{2}\Lambda_\sigma^{-1/2}||u'||_g||Du||_g+A_1, \tag{9.66}$$

Since $\tau<0$, it will hold that

$$\frac{dE_\alpha}{dt}-k\tau E_\alpha\leq|\tau A|,\quad A\equiv A_1+A_2+A_3. \tag{9.67}$$

if the quadratic form

$$Q_{\alpha,k}(x)\equiv(1-2\alpha-\frac{k}{2})x_0^2+(2\alpha-\frac{k}{2})x_1^2-\alpha(1-k)\sqrt{2}\Lambda_\sigma^{-1/2}x_0x_1 \tag{9.68}$$

is non negative.

The quadratic form $Q_{\alpha,k}$ is non negative if:

$$k\leq 4\alpha,\quad\text{and}\quad k\leq 2(1-2\alpha) \tag{9.69}$$

and k is such that its discriminant is negative, that is:

$$2\alpha^2\Lambda_\sigma^{-1}(1-k)^2-4(2\alpha-\frac{k}{2})(1-2\alpha-\frac{k}{2})\leq 0 \tag{9.70}$$

The inequalities 9.69 imply

$$k\leq 1, \tag{9.71}$$

The inequality 9.70 reads

$$(1-2\Lambda_\sigma^{-1}\alpha^2)k^2-(1-2\Lambda_\sigma^{-1}\alpha^2)2k-2\Lambda_\sigma^{-1}\alpha^2+8\alpha(1-2\alpha)>0 \tag{9.72}$$

We have already supposed that $1-2\Lambda_\sigma^{-1}\alpha^2>0$, the inequality above is therefore equivalent to:

$$k^2-2k+1-\frac{(1-4\alpha)^2}{(1-2\Lambda_\sigma^{-1}\alpha^2)}>0 \tag{9.73}$$

that is

$$k<1-\frac{1-4\alpha}{(1-2\Lambda_\sigma^{-1}\alpha^2)^{\frac{1}{2}}} \tag{9.74}$$

There will exist such a $k > 0$ if

$$\frac{1-4\alpha}{(1-2\Lambda_\sigma^{-1}\alpha^2)^{\frac{1}{2}}} < 1 \tag{9.75}$$

Since $\alpha > 0$ this inequality reduces to:

$$-2\Lambda_\sigma^{-1}\alpha - 16\alpha + 8 > 0. \tag{9.76}$$

i.e.

$$\alpha < \frac{4}{8+\Lambda_\sigma^{-1}} \tag{9.77}$$

We remark that this inequality imposes the hypothesis first made on α, since elementary calculus shows that, for any Λ, it holds that:

$$\frac{\Lambda^{\frac{1}{2}}}{\sqrt{2}} \leq \frac{4}{8+\Lambda^{-1}}, \tag{9.78}$$

the inequality being attained only for $\Lambda = \frac{1}{8}$.

We distinguish two cases

1. $\Lambda_\sigma > \frac{1}{8}$. In this case it is possible to take $\alpha = \frac{1}{4},\ k = 1$ and obtain immediately

$$\frac{dE_{\frac{1}{4}}}{dt} - \tau E_{\frac{1}{4}} \leq |\tau A|. \tag{9.79}$$

2. $\Lambda_\sigma \leq \frac{1}{8}$. We have then:

$$\frac{4}{8+\Lambda_\sigma^{-1}} \leq \frac{1}{4}. \tag{9.80}$$

We choose α such that it satisfies the inequality 9.77, which implies in this case $\alpha < \frac{1}{4}$, and then $k > 0$ such that it satisfies 9.74. ∎

10 Corrected second energy.

We define a **corrected second energy** $E_\alpha^{(1)}$ by the formula

$$E_\alpha^{(1)}(t) = E^{(1)}(t) + \alpha\tau E_c^{(1)}(t), \quad E_c^{(1)}(t) \equiv \int_{\Sigma_t} \hat{\Delta}_g u.u' \mu_g \tag{10.1}$$

10.1 Lower bound.

We have, according to previous notations,

$$\hat{\Delta}_g u.u' \equiv 2\Delta_g \gamma \gamma' + \frac{1}{4} e^{-4\gamma} \Delta_g \omega \omega' + b_1 \tag{10.2}$$

with

$$b_1 \equiv e^{-4\gamma} g^{ab} (\partial_a \omega \partial_b \omega \gamma' - \partial_a \gamma \partial_b \omega \omega'). \tag{10.3}$$

hence, using the lemma 5.2:

$$B_1 \equiv |\int_{\Sigma_t} b_1 \mu_g| \leq |\tau| C_{E,\lambda,\sigma} (\varepsilon + \varepsilon_1)^3. \tag{10.4}$$

The Cauchy Schwarz inequality and the Poincaré inequality ($\bar{\gamma}'$ is a constant on Σ_t and on a compact manifold $\int_{\Sigma_t} \Delta_g \gamma \mu_g = 0$) give that:

$$|\int_{\Sigma_t} \Delta_g \gamma \gamma' \mu_g| = |\int_{\Sigma_t} \Delta_g \gamma (\gamma' - \bar{\gamma}') \mu_g| \leq e^{\lambda_M} \Lambda_\sigma^{-1/2} ||\Delta_g \gamma||_g ||D\gamma'|| \tag{10.5}$$

while

$$\begin{aligned} |\int_{\Sigma_t} (\Delta_g \omega) e^{-4\gamma} \omega' \mu_g| &= |\int_{\Sigma_t} \Delta_g \omega (e^{-4\gamma} \omega' - \overline{e^{-4\gamma} \omega'}) \mu_g| \\ &\leq e^{\lambda_M} \Lambda_\sigma^{-1/2} ||\Delta_g \omega||_g ||D(e^{-4\gamma} \omega')|| \end{aligned}$$

It holds that

$$||D(e^{-4\gamma}\omega')|| = ||e^{-4\gamma}(D\omega' - 4D\gamma\omega')|| \tag{10.6}$$

$$\leq e^{-2\gamma_m} (||e^{-2\gamma} D\omega')|| + 4||e^{-2\gamma}\omega'||_4 ||D\gamma||_4 \tag{10.7}$$

while

$$||\Delta_g \omega||_g \leq e^{2\gamma_M} ||e^{-2\gamma} \Delta_g \omega||_g \tag{10.8}$$

Using the bound (lemma 9.1) of $\gamma_M - \gamma_m$ and the inequalities on the L^4 norms $||.||_4$ (lemma 5.1), we find an inequality of the form:

$$|\int_{\Sigma_t} (\Delta_g \omega) e^{-4\gamma} \omega' \mu_g|| \leq e^{\lambda_M} \Lambda_\sigma^{-1/2} ||e^{-2\gamma} \Delta_g \omega||_g ||e^{-2\gamma} D\omega')|| + B_2$$

where B_2 satisfies an inequality of the same type as B_1. We have shown that

$$|\int_{\Sigma_t} \Delta_g u.u' \mu_g| \leq e^{\lambda_M} \Lambda_\sigma^{-1/2} ||\Delta_g u||_g ||Du'|| + B_1 + B_2.$$

The estimates of the lemma 5.1 and the inequalities

$$||\Delta_g u||_g \leq ||\hat{\Delta}_g u||_g + || \, |Du|^2||_g, \quad ||Du'|| \leq ||\hat{\nabla} u'|| + || \, |Du|^2||_g^{\frac{1}{2}} || \, |u'|^2||_g^{\frac{1}{2}} \tag{10.9}$$

give

$$e^{\lambda_M} \Lambda_\sigma^{-1/2} ||\Delta_g u||_g ||Du'|| \leq e^{\lambda_M} \Lambda_\sigma^{-1/2} ||\hat{\Delta}_g u||_g ||\hat{\nabla} u'|| + B_3$$

with

$$B_3 \leq |\tau| C_{E,\lambda,\sigma} (\varepsilon + \varepsilon_1)^3. \tag{10.10}$$

Using the estimate 6.23 of $e^{\lambda_M}|\tau| - \sqrt{2}$ we have:

$$|\int_{\Sigma_t} \Delta_g u.u' \mu_g| \leq \sqrt{2} |\tau|^{-1} \Lambda_\sigma^{-1/2} ||\hat{\Delta}_g u||_g ||\hat{\nabla} u'|| + B_1 + B_2 + B_3 + B_4. \tag{10.11}$$

with

$$B_4 = |e^{\lambda_M} - \sqrt{2}|\tau|^{-1}|\Lambda_\sigma^{-1/2} ||\hat{\Delta}_g u||_g ||\hat{\nabla} u'|| \leq |\tau| C_\sigma (\varepsilon + \varepsilon_1)^3$$

We deduce from these estimates, with $Q_{\alpha,\Lambda}$ the same quadratic form as in 9.18 but with $y_1 \equiv |\tau|^{-1} ||\hat{\Delta}_g u||_g$, $y_0 \equiv |\tau|^{-1} ||\hat{\nabla} u'||$, that:

$$\tau^{-2} E_\alpha^{(1)}(t) \geq Q_{\alpha,\Lambda}(y_0, y_1) - C_{E,\sigma,\lambda} (\varepsilon + \varepsilon_1)^3 \tag{10.12}$$

Theorem 10.1 *If α is chosen satisfying 9.19 there exists a number $\eta_4 > 0$ and $L > 0$ such that*

$$E_\alpha + \tau^{-2} E_\alpha^{(1)} \geq L(\varepsilon^2 + \varepsilon_1^2) \tag{10.13}$$

as soon as

$$\varepsilon + \varepsilon_1 \leq \eta_4. \tag{10.14}$$

PROOF. We have found that

$$\psi(t) \equiv E_\alpha(t) + \tau^{-2} E_\alpha^{(1)}(t) \geq Q_{\alpha,\Lambda}(y, x) - (A + B) \tag{10.15}$$

where $Q_{\alpha,\Lambda}(x, y)$ is the quadratic form

$$Q_{\alpha,\Lambda}(x, y) \equiv Q_{\alpha,\Lambda}(x) + Q_{\alpha,\Lambda}(y). \tag{10.16}$$

and $A + B$ admits a bound of the form

$$|A + B| \leq C_{E,\sigma,\lambda} (\varepsilon^2 + \varepsilon_1^2)^{\frac{3}{2}}. \tag{10.17}$$

We have

$$Q_{\alpha,\Lambda}(x,y) > K(\varepsilon^2+\varepsilon_1^2) \equiv \frac{1}{2}K(x_0^2+x_1^2+y_0^2+y_1^2) \tag{10.18}$$

if the quadratic form Q_K defined in the section 9.1 is positive definite. The conditions on α and the corresponding limitation on K as the same as in the section 9.1, and the proof continues along the same line. ∎

10.2 Decay of the second corrected energy.

We have (recall $\frac{d\tau}{dt} = \tau^2$) :

$$\frac{dE_\alpha^{(1)}}{dt} \equiv \frac{dE^{(1)}}{dt} + \alpha\tau\mathcal{R}^{(1)} \quad \mathcal{R}^{(1)} \equiv \frac{d}{dt}E_c^{(1)} + \tau E_c^{(1)}$$

that is:

$$\mathcal{R}^{(1)} = \int_{\Sigma_t} \{\hat{\partial}_0\hat{\Delta}_g u.u' + \hat{\Delta}_g u.(\hat{\partial}_0 u' - N\tau.u' + \tau u')\}\mu_g. \tag{10.19}$$

We have found in lemma 4.1 that

$$\hat{\partial}_0\hat{\Delta}_g u^A \equiv g^{ab}\hat{\nabla}_a\hat{\partial}_0\partial_b u^A + N\tau\hat{\Delta}_g u^A + F_1^A + \hat{F}_1^A, \tag{10.20}$$

with

$$\hat{F}_1^A \equiv g^{ab}R_{CB}{}^A{}_D\partial_0 u^C\partial_a u^B\partial_b u^D$$

and

$$F_1^A \equiv 2\partial_c u^A(h_g^{ac}\partial_a N + N\nabla_a k^{ac}) + 2Nh_g^{ab}\hat{\nabla}_a\partial_b u^A \tag{10.21}$$

that is, using the equation

$${}^{(3)}R_0^c \equiv -N\nabla_a k^{ac} = \partial_0 u.\partial^c u$$

$$F_1^A \equiv 2\partial_c u^A(h_g^{ac}\partial_a N - \partial_0 u.\partial^c u) + 2Nh_g^{ab}\hat{\nabla}_a\partial_b u^A. \tag{10.22}$$

Partial integration gives, using also the identity $\hat{\partial}_0\partial_b u \equiv \hat{\nabla}_b\partial_0 u \equiv \hat{\nabla}_b(Nu')$,

$$\int_{\Sigma_t}(\hat{\partial}_0\hat{\Delta}_g u).u'\mu_g = \int_{\Sigma_t}\{-N|\hat{\nabla}u'|_g^2 - \partial^a N\hat{\nabla}_a u'.u' + N\tau\hat{\Delta}_g u.u' + (F_1+\hat{F}_1).u'\}\mu_g \tag{10.23}$$

On the other hand, if u satisfies the equation

$$-\hat{\partial}_0 u' + \hat{\nabla}^a(N\partial_a u) + \tau N u' = 0 \tag{10.24}$$

it holds that:

$$\int_{\Sigma_t} \hat{\Delta}_g u.(\hat{\partial}_o u' - \tau N u')\mu_g = \int_{\Sigma_t} \{N|\hat{\Delta}_g u|^2 + \partial^a N \partial_a u.\hat{\Delta}_g u\}\mu_g$$

We have found that:

$$\mathcal{R}^{(1)} = \int_{\Sigma_t} \{-N|\hat{D}u'|^2 + N|\hat{\Delta}_g u|^2 + \tau(N+1)\hat{\Delta}_g u.u'\}\mu_g + \tilde{\mathcal{R}}^{(1)} \tag{10.25}$$

with

$$\tilde{\mathcal{R}}^{(1)} \equiv \int_{\Sigma_t} \{-\partial^a N \hat{\nabla}_a u'.u' + (F_1 + \hat{F}_1).u' + \partial^a N \partial_a u.\hat{\Delta}_g u\}\mu_g. \tag{10.26}$$

Using the expression of $\frac{dE^{(1)}}{dt}$ we find that:

$$\frac{dE^{(1)}_\alpha}{dt} = \tau \int_{\Sigma_t} \{N(2-2\alpha)J_0 + N(2\alpha+1)J_1 + (N+1)\alpha\tau\hat{\Delta}_g u.u']\}\mu_g + Z_1 + \alpha\tau\tilde{\mathcal{R}}^{(1)}$$

which implies:

$$\frac{dE^{(1)}_\alpha}{dt} - (2+k)\tau E^{(1)}_\alpha = \tau\{\int_{\Sigma_t} \{(2N-2-k-2\alpha N)J_0 + (2\alpha N + N - 2 - k)J_1$$
$$+\alpha\tau(N+1-2-k)\hat{\Delta}_g u.u'\}\mu_g + Z_1 + \alpha\tau\tilde{\mathcal{R}}^{(1)} \tag{10.27}$$

which we write:

$$\frac{dE^{(1)}_\alpha}{dt} - (2+k)\tau E^{(1)}_\alpha \leq \tau\{\int_{\Sigma_t} \{(2-k-4\alpha)J_0 + (4\alpha - k)J_1$$
$$+\alpha\tau(1-k)\hat{\Delta}_g u.u'\}\mu_g + Z_1 + \alpha\tau\tilde{\mathcal{R}}^{(1)} + Z_2. \tag{10.28}$$

with

$$Z_2 \equiv \tau \int_{\Sigma_t} \{(N-2)(2-2\alpha)J_0 + (N-2)(2\alpha+1)J_1 + (N-2)\alpha\tau\hat{\Delta}_g u.u']\}\mu_g. \tag{10.29}$$

We have found in section 8.2 that

$$|Z_1| \le |\tau|^3 C_{E,\lambda,\sigma}(\varepsilon+\varepsilon_1)^3 \tag{10.30}$$

It results immediately from the estimate of $N-2$, section 6.2, that Z_2 satisfies an inequality of the same type.

Some terms of $\tilde{\mathcal{R}}^{(1)}$ are bounded using the $L^\infty(g)$ estimate 6.6 of DN, which gives that

$$|\int_{\Sigma_t}\{-\partial^a N\hat{\nabla}_a u'.u' + \partial^a N\partial_a u.\hat{\Delta}_g u\}\mu_g| \le \tau^2 C_{E,\sigma,\lambda}\varepsilon^2\varepsilon_1(\varepsilon+\varepsilon_1). \tag{10.31}$$

The estimate of the remaining ones uses similar techniques as those of section 9 and lead to an inequality of the form

$$|\tilde{\mathcal{R}}^{(1)}| \le \tau^2 C_{E,\sigma,\lambda}(\varepsilon+\varepsilon_1)^4. \tag{10.32}$$

Theorem 10.2 *Under the conditions on α and k given in the theorem 9.2 the following inequality holds:*

$$\frac{dE_\alpha^{(1)}}{dt} - (2+k)\tau E_\alpha^{(1)} \le |\tau|^3 B \tag{10.33}$$

with

$$|B| \le C_{E,\lambda,\sigma}(\varepsilon+\varepsilon_1)^3.$$

Proof. We have seen that (10.11)

$$|\int_{\Sigma_t}\tau\hat{\Delta}_g u.u'\mu_g| \le \sqrt{2}\Lambda_\sigma^{-\frac{1}{2}}||\hat{\Delta}_g u||_g\ ||\hat{\nabla}u'||_g + Z_3 \tag{10.34}$$

with

$$|Z_3| \le \tau^2 C_{E,\sigma,\lambda}(\varepsilon+\varepsilon_1)^3. \tag{10.35}$$

Therefore we deduce from 10.28 and the definition of y_0, y_1 that

$$\frac{dE_\alpha^{(1)}}{dt} - (2+k)\tau E_\alpha^{(1)} \le \tau^3 Q_{\alpha,k}^{(1)}(y) + |\tau|^3 B$$

with

$$B \equiv Z_1 + \alpha\tau\tilde{\mathcal{R}}^{(1)} + Z_2 + |\tau|\alpha Z_3, \quad |B| \le C_{E,\sigma,\lambda}(\varepsilon^2+\varepsilon_1^2)^{\frac{3}{2}} \tag{10.36}$$

and

$$Q_{\alpha,k}^{(1)}(y) \equiv (1-\frac{k}{2}-2\alpha)y_0^2 + (2\alpha-\frac{k}{2})y_1^2 + \sqrt{2}\Lambda_\sigma^{-\frac{1}{2}}\alpha\tau(1-k)y_0y_1. \tag{10.37}$$

This quadratic form in y is non negative under the same conditions as the form $Q_{\alpha,k}(x)$. The conclusion follows, since $\tau < 0$. ■

11 Decay of the total energy.

We make the following a priori hypothesis, for all $t \geq t_0$ for which the considered quantities exist

- **Hypothesis** $\mathbf{H}_\sigma$: 1. The t dependent numbers C_σ are uniformly bounded by a constant M.

2. There exist $\Lambda > 0$ such that $\Lambda_\sigma \geq \Lambda$.

- We choose α such that

$$\alpha = \frac{1}{4} \quad \text{if} \quad \Lambda > \frac{1}{8}, \quad \alpha < \frac{4}{8 + \Lambda^{-1}} \leq \frac{1}{4} \quad \text{if} \quad \Lambda \leq \frac{1}{8}. \tag{11.1}$$

- **Hypotheses** $\mathbf{H}_E^\eta$: The t dependent energies ε^2 and ε_1^2 having been supposed bounded by a number c_E we suppose, moreover that they satisfy the inequalities 5.25, 6.3, 9.25, 10.14.

  We have seen (theorem 5.7) that the hypothesis H_λ is then satisfied.

We denote by M_i any given positive number dependent on the bounds of these H's hypothesis but independent of t.

We have defined $\psi(t)$ to be the **total corrected energy** namely:

$$\psi(t) \equiv E_\alpha(t) + \tau^{-2} E_\alpha^{(1)}$$

We have seen (10.13) that $\psi(t)$ bounds the total energy $\phi(t) \equiv \varepsilon^2 + \varepsilon_1^2$ by an inequality of the form:

$$\phi(t) \equiv \varepsilon^2 + \varepsilon_1^2 \leq M_0 \psi(t), \quad M_0 = L^{-1}. \tag{11.2}$$

Lemma 11.1 *Under the hypotheses H_σ and H_E^η the function ψ satisfies a differential inequality of the form*

$$\frac{d\psi}{dt} \leq -\frac{k}{t}(\psi - M_1 \psi^{3/2}) \tag{11.3}$$

PROOF. The inequalities 9.56 and 10.33 together with the choice $\tau = -\frac{1}{t}$, and the bound 10.13. ∎

Theorem 11.2 *Under the hypotheses H_σ and H_E^η there exists a number $k > 0$ such that the total energy $E_{tot}(t) \equiv \phi(t) \equiv \varepsilon^2 + \varepsilon_1^2$ satisfies an estimate of the form*

$$t^k \phi(t) \leq M_2 \phi(t_0) \tag{11.4}$$

if it is small enough initially.

Proof. We suppose that $\psi_0 \equiv \psi(t_0)$ satisfies

$$\psi_0^{1/2} < \frac{1}{M_1} \tag{11.5}$$

Then ψ starts decreasing, continues to decrease as long as it exists, therefore $(\psi - M_1\psi^{3/2}) > 0$ and the inequality 11.3 is equivalent to

$$\frac{dz}{z - M_1 z^3} + \frac{k}{2}\frac{dt}{t} \leq 0, \quad \text{with} \quad \psi = z^2.$$

This inequality gives by integration:

$$\log\{\frac{z}{(1 - M_1 z)z_0}\frac{(1 - M_1 z_0)}{z_0}\} + \frac{1}{2}k \log \frac{t}{t_0} \leq 0$$

equivalently

$$\{\frac{z}{(1 - M_1 z)}\frac{(1 - M_1 z_0)}{z_0}\}\{\frac{t}{t_0}\}^{\frac{1}{2}k} \leq 1 \tag{11.6}$$

and, a fortiori,

$$t^k \psi \leq \frac{t_0^k \psi_0}{(1 - M_1 z_0)^2}. \tag{11.7}$$

Hence, using 11.2, the **decay estimate**

$$t^k \phi(t) \leq M_2 \phi_0. \tag{11.8}$$

■

12 Teichmüller parameters.

Instead of considering as in [CB-M1], [CB-M2] the Dirichlet energy of the metric σ we use directly the estimate 7.7 of dQ/dt which we now write, using 11.8:

$$|\frac{dQ}{dt}| \leq C_{\sigma,E} t^{-(1+\frac{k}{2})} \phi(t_0)^{1/2} \tag{12.1}$$

Therefore:

Theorem 12.1 *There exists M_3 such that*

$$|Q(t) - Q(t_0)| \leq M_3 \phi(t_0)^{1/2}. \tag{12.2}$$

13 Global existence.

Theorem 13.1 *Let $(\sigma_0, q_0) \in C^\infty(\Sigma_0)$ and $(u_0, \dot{u}_0) \in H_2(\Sigma_0, \sigma_0) \times H_1(\Sigma_0, \sigma_0)$ be initial data for the Einstein equations with U(1) isometry group on the initial manifold $M_0 \equiv \Sigma_0 \times U(1)$, with Σ_0 compact, orientable and of genus greater than one, σ_0 chosen such that $R(\sigma_0) = -1$. Suppose the initial integral condition 2.1 (with $n = 0$) satisfied. Then there exists a number $\eta_0 > 0$ such that if*

$$\phi(t_0) \equiv E_{tot}(t_0) < \eta_0 \tag{13.1}$$

these Einstein equations have a solution on $M \times [t_0, \infty)$, with initial values determined by $\sigma_0, q_0, u_0, \dot{u}_0$. The parameter t is $t = -\tau^{-1}$, with τ the mean extrinsic curvature of $\Sigma \times \{t\}$ in the lorentzian metric ${}^{(3)}g$ on $\Sigma \times [t_0, \infty)$.

This solution is unique[9] up to the choice of a section of Tecichmuller space and a gauge choice for A.

Proof. We first prove that $E_{tot}(t)$ is uniformly bounded, and decays to zero (without a priori hypothesis). We have obtained in the previous sections, under the hypotheses H_E^η and H_σ, the following result: there are numbers M_i depending only on c_E and c_σ such that

$$t^k E_{tot}(t) \leq M_2 E_{tot}(t_0) \tag{13.2}$$

and

$$|Q(t) - Q(t_0)| \leq M_3 \phi(t_0)^{1/2}. \tag{13.3}$$

Now consider the pair of t dependent numbers

$$(\phi(t), \zeta(t)), \quad \zeta(t) \equiv |Q(t) - Q(t_0)|$$

The inequalities 13.3, 13.4 show that the hypothesis (where c_E satisfies H_E^η)

$$\phi(t) \leq c_E, \ \zeta(t) \leq c_\sigma$$

[9]The global uniqueness theorem of CB-Geroch says that it is geometrically unique in the class of globally hyperbolic spacetimes.

imply that there exists $\eta_0 > 0$ such that $\phi(t_0) \leq \eta_0$ implies that the pair belongs to the subset $U_1 \subset R^2$ defined by the inequalities:

$$U_1 \equiv \{ \ \phi(t) < c_E, \ \zeta(t) < c_\sigma \}.$$

Therefore for such an η_0 the pair belongs either to U_1 or to the subset U_2 defined by

$$U_2 \equiv \{ \ \phi(t) > c_E \quad \text{or} \quad \zeta(t) > c_\sigma \}$$

These subsets are disjoint. We have supposed that for $t = t_0$ it holds that $(\phi(t_0), \zeta(t_0)) \in U_1$ hence, by continuity in t, $(\ \phi(t), \ \zeta(t)) \in U_1$ for all t.

We have now proved that the total energy is uniformly bounded, and σ_t uniformly equivalent to σ_0.

To complete the proof of existence of the spacetime for $t \in [t_0, \infty)$ we need the following lemma.

Lemma. The H_2 norm of the pair of scalar functions (γ, ω) is uniformly bounded, as well of the H_1 norm of $(\partial_t\gamma, \partial_t\omega)$.

Proof of lemma. We have already proven in section 9 the uniform bound of $||D\gamma||$ and $||D^2\gamma||$ in terms of the total energy. On the other hand it holds that

$$\gamma - \gamma_0 = \int_{t_0}^{t} \partial_t\gamma dt \tag{13.4}$$

hence

$$||\gamma - \gamma_0|| \leq \int_{t_0}^{t} ||\partial_t\gamma|| dt \tag{13.5}$$

Using previous estimates, the fall off of the energy and the property

$$||\partial_t\gamma|| \leq e^{-\lambda_m} ||\partial_t\gamma||_g \tag{13.6}$$

we find that there exists a number M such that

$$||\gamma - \gamma_0|| \leq M \int_{t_0}^{t} t^{-(1+k)} dt, \tag{13.7}$$

which completes the proof of the uniform bound of $||\gamma||_{H_2}$, hence also of γ in C^0.

When γ_M is uniformly bounded one can bound $||D\omega|| \leq e^{2\gamma_M} ||e^{-2\gamma} D\omega||$ with the first energy and $||D^2\omega||$ with the total energy, in a manner analogous as the one used for $||D^2\gamma||$. We just recalled the estimate of $||\partial_t\gamma||$, the estimate of $||\partial_t\omega||$ is similar, when γ has been bounded. It is also easy to bound $||D\partial_t\omega||$ and $||D\partial_t\gamma||$. ■

Corollary 13.2 *1. This solution is globally hyperbolic, future timelike and null complete.*

2. It is asymptotic to a flat solution:

$$^{(4)}g = -4dt^2 + 2t^2\sigma_\infty + \theta_\infty^2 \tag{13.8}$$

with σ_∞ a metric on Σ independent of t and of scalar curvature -1, and θ_∞ a 1-form on $\Sigma \times S^1$ of the type

$$\theta_\infty = C(dx^3 + H), \tag{13.9}$$

where C is a constant and H is a harmonic 1-form on (Σ, σ_0).

Proof. 1. The orthogonal trajectories to the space sections $M \times \{t\}$ have an infinite proper length since the lapse N is bounded below by a strictly positive number. It can be checked that the conditions given in C.B and Cotsakis for global hyperbolicity, and for future and null completeness are satisfied by $(\Sigma \times R, ^{(3)}g)$.

2. The theorem 5.7 and the decay of $\varepsilon + \varepsilon_1$ show that λ tends to $2t^2$ in C^0 norm when t tends to infinity.

The decay estimate of $\frac{dQ}{dt}$ show that Q tends to a point Q_∞ in T_{eich} when t tends to infinity, σ tends to $\sigma(Q_\infty)$.

The lapse and shift estimates 6.2 and 7.9, 7.11 show that N tends to 2 and ν tends to zero in C^0 norm when t tends to infinity.

The integral formula for γ shows that $\gamma(t,.) - \gamma_0(.)$ tends to a function on Σ, $\hat{\gamma}_\infty(.)$, in L^2 norm when t tends to infinity, hence γ tends to $\gamma_\infty = \gamma_0 + \hat{\gamma}_\infty$ in this norm, therefore a fortiori $\gamma(t,.)$ tends to $\gamma_\infty(.)$ in the sense of distributions on Σ. We know on the other hand that $||D\gamma||$ tends to zero, hence $D\gamma$ tends to zero in the sense of distributions. Since derivation in this sense is a continuous operator it holds that $D\gamma_\infty = 0$, therefore γ_∞ is a constant.

An analogous reasonning holds for ω. The value of ω_∞ does not appear in the expression of F.

The estimates of section 2.2 of $\hat{A}$ and A_t (in Coulomb gauge) show that they both tend to zero in C^0 norm on Σ. The differential formula giving the $c_i(t)$ shows then that the 1 form $\tilde{A}$ tends in C^0 norm to the harmonic form on Σ, $H_\infty = c_{i,\infty}H_{(i)}$The spacetime metric is asymptotic to the metric

$$^{(4)}g = e^{-2\gamma_\infty}(-4dt^2 + 2t^2\sigma_\infty) + e^{2\gamma_\infty}(dx^3 + H_\infty)^2 \tag{13.10}$$

which takes the indicated form by rescaling of t. ■

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