

The Cauchy problem on spacetimes that are not globally hyperbolic

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Abstract

The initial value problem is well-defined on a class of spacetimes broader than the globally hyperbolic geometries for which existence and uniqueness theorems are traditionally proved. Simple examples are the time-nonorientable spacetimes whose orientable double cover is globally hyperbolic. These spacetimes have generalized Cauchy surfaces on which smooth initial data sets yield unique solutions. A more difficult problem is to characterize the class of spacetimes with closed timelike curves that admit a well-posed initial value problem. Examples of spacetimes with closed timelike curves are given for which smooth initial data at past null infinity has been recently shown to yield solutions. These solutions appear to be unique, and uniqueness has been proved in particular cases. Other examples, however, show that confining closed timelike curves to compact regions is not sufficient to guarantee uniqueness. An approach to the characterization problem is suggested by the behavior of congruences of null rays. Interacting fields have not yet been studied, but particle models suggest that uniqueness (and possibly existence) is likely to be lost as the strength of the interaction increases.

I. INTRODUCTION

Motivating the definition of a *globally hyperbolic* spacetime are two facts: On globally hyperbolic spacetimes, wave equations have a well-defined initial value formulation; and the ordinary causal structure of a globally hyperbolic spacetime mirrors the ordinary causal structure observed in the universe. The requirement that the initial value problem be well defined, however, picks out a broader class of spacetimes. And the causal structure of the physical universe on the largest and smallest scales may similarly not conform to that of a globally hyperbolic spacetime.

In particular, a class of spacetimes that are not globally hyperbolic and nevertheless admit a well-defined initial value problem are Lorentzian 4-geometries with a single spacelike boundary – Lorentzian *universes-from-nothing* (Friedman and Higuchi [1, 2]). These are the metrics that arise in a Lorentzian path-integral construction of the Hartle-Hawking wavefunction of the Universe. There is large class of such geometries, spacetimes that are compact on one side of a spacelike boundary. A simple two-dimensional example, considered in more detail below, is a Möbius strip with a flat Lorentz metric, for which the direction orthogonal to the median circle is timelike. On each underlying manifold of such a spacetime, one can choose metrics that have no closed timelike curves and for which the boundary remains spacelike; time nonorientability is then their only causal pathology. With metrics so chosen, these spacetimes provide the only examples of topology change in which one has a smooth, nondegenerate Lorentzian metric without closed timelike curves. Instead, the spacetimes are time nonorientable. The initial value problem for these spacetimes is discussed in Sect. II.

The more difficult question is the class of spacetimes that have closed timelike curves (CTCs) and that nevertheless allow a well-defined initial value problem for hyperbolic systems. This brief review outlines recent work that has been done in proving existence and uniqueness of solutions to the scalar wave equation on a class of spacetimes with closed timelike curves. Sect. III introduces the subject with several two-dimensional examples that are easily analyzed and illustrate the obstacles to the existence of a generalized initial value problem. These obstacles are less severe in four dimensions, and Sect. IV considers four dimensional examples in which existence and uniqueness theorems have been proved. These examples are quite restricted; Section V outlines the heuristic arguments that have been made for the existence of a well-defined initial value problem in a much broader class of spacetimes. The section concludes with two conjectures that would partly characterize such a class.

Let us first generalize the definition of a Cauchy surface to allow spacetimes that are not globally hyperbolic. Let M, g be a smooth spacetime.

Definition. A *generalized Cauchy surface* Σ is an achronal hypersurface of M for which the initial value problem for the scalar wave equation is well-defined:

For any smooth data in $L_2(\Sigma)$ with finite energy, for a scalar field Φ , there is a unique solution Φ on M .

One can then ask what the class is of spacetimes for which the initial value problem is well-defined.

To see that this class is larger than the class of globally hyperbolic spacetimes, we begin with the time non-orientable spacetimes mentioned above.

II. TIME-NONORIENTABLE SPACETIMES WITH A WELL-DEFINED INITIAL VALUE PROBLEM

We begin with a two dimensional example and then describe a general class of geometries on a countably infinite set of 4-manifolds. Consider the Möbius strip M with a metric for which its median circle is spacelike. One can construct such a 2-dimensional spacetime from a cylinder $\bar{M} = \mathbb{R} \times S^1$, with an obvious choice of Minkowski metric that makes copies of \mathbb{R} into timelike lines orthogonal to copies of S^1 . In terms of the natural chart t, ϕ , on \bar{M} , the metric is $-dt^2 + a^2 d\phi^2$, some length a . To construct the Möbius strip, identify each point (t, ϕ) , with its antipodal point

$$A(t, \phi) = (-t, \phi + \pi). \quad (1)$$

The Möbius strip is the quotient space $M = \bar{M}/A$; because A is an isometry, M inherits a flat Lorentzian metric for which its median circle Σ (the image of the circle $t = 0$) is spacelike.

It is easy to see that the median circle Σ is a generalized Cauchy surface (as is any boosted image of it). Initial data is a pair $\Phi, \nabla\Phi$ (with ∇ the 2-dimensional gradient) on Σ . Data on Σ lifts to initial data on a Cauchy surface $\bar{\Sigma}$ of the cylinder, and the data is antipodally symmetric. Because the the cylinder is globally hyperbolic, there is a unique solution $\bar{\Phi}$ to the wave equation with this data, and that solution is itself antipodally symmetric. Thus there is a field Φ on the Möbius strip, whose lift to the cylinder is $\bar{\Phi}$; and Φ is the unique solution to the wave equation on M with the specified initial data.

In this flat example, the Möbius strip has closed timelike curves (CTCs) *and* is time norientable. If one chooses a deSitter metric on the cylinder instead of the flat metric, the antipodal map remains an isometry, and the Möbius strip inherits a local deSitter metric, a metric for which it has no CTCs. (CTCs arise from timelike lines that emerge in opposite directions - forward and backwards with respect to a locally defined time direction; in the deSitter geometry circles far from the median circle are large, and they expand fast enough that the timelike lines never meet.) In both the flat and the deSitter case (i.e., with and without CTCs), because the orientable double-cover of M, g is globally hyperbolic, the spacetime has a generalized Cauchy surface.

More generally let $\bar{\Sigma}$ be any 3-manifold that admits a free involution, a diffeo I that has no fixed points. There are countably many spherical spaces and countably many hyperbolic spaces that admit such involutions. As in the above construction, one defines on a cylinder $\bar{M} = \mathbb{R} \times \bar{\Sigma}$ an antipodal map $A = T \times I$, where $T : \mathbb{R} \rightarrow \mathbb{R}$ is time reversal:

$$A(t, p) = (-t, I(p)). \quad (2)$$

A is again a free involution, and the manifold of the spacetime is again the quotient

$$M = \bar{M}/A. \quad (3)$$

We will choose a metric for which the 3-manifold $\Sigma = \{0\} \times \bar{\Sigma}/A$ is a generalized Cauchy surface. Any metric g on M , for which each of the surfaces corresponding to $\{t\} \times \bar{\Sigma}$ are spacelike, will do. The pullback of g to \bar{M} is a metric for which M is foliated by the spacelike hypersurfaces $\{t\} \times \bar{\Sigma}$ and is therefore globally hyperbolic. (For example, let 3g be any Riemannian metric on Σ , ${}^3\bar{g}$ its pullback to \bar{M} . The metric $-dt^2 + {}^3\bar{g}$ on \bar{M} is antipodally symmetric and induces an suitable metric on M .)

By our construction of \bar{g} , the antipodal map A is an isometry. Again, the lift to $\bar{\Sigma}$ of data $\Phi, \nabla\Phi$ on Σ is an initial data set $\bar{\Phi}, \nabla\bar{\Phi}$ that is antipodally symmetric (invariant under A). Because \bar{M}, \bar{g} is foliated by globally hyperbolic, this data has a unique time evolution, $\bar{\Phi}$; because both the data and the spacetime \bar{M}, \bar{g} are antipodally symmetric, $\bar{\Phi}$ is antipodally symmetric. The field $\bar{\Phi}$ is therefore the lift to \bar{M} of a solution Φ to the scalar wave equation on M . Finally, Φ is unique because $\bar{\Phi}$ is unique.

Classically, spacetimes of the kind considered in this section (nonorientable spacetimes whose orientable double cover is globally hyperbolic), are locally indistinguishable from their covering spacetimes. Treatments of classical spinor fields and of quantum field theory on such spacetimes are given in Refs. [1, 3, 4]

III. TWO-DIMENSIONAL SPACETIMES WITH CLOSED TIMELIKE CURVES

On spacetimes with CTCs, the initial value problem is more subtle. Simple examples show that some spacetimes with CTCs have a generalized Cauchy surface; but the Cauchy problem is not well-defined in generic two-dimensional spacetimes, and other examples in this section illustrate several essentially different ways in which CTCs can block the existence of smooth solutions or allow more than one solution for the same initial data on a spacelike surface.

We again begin with two-dimensional spacetimes, built from Minkowski space. Obstacles are most severe here, and we will see that some can be overcome in higher dimensions. First a familiar case in which the one's naive expectation of the way CTCs prevent solutions is fulfilled. Identify the edges of the strip of Minkowski space between two parallel, straight timelike lines, $t = 0$ and $t = T$:

$$(t = 0, x) \equiv (t = T, x). \quad (4)$$

Here the only candidates for an initial value surface are spacelike lines Σ extending to spatial infinity. Minkowski space \bar{M} covers this spacetime, and data on any one of the spacelike lines of \bar{M} that covers Σ can be uniquely evolved in the covering space, but the resulting solution corresponds to a solution on the

original spacetime only if it is suitably periodic. It must have the same value on each covering line $\bar{\Sigma}$; and almost no data yields such a solution. This is essentially the grandfather paradox: Locally one can construct a unique solution on M ; but when extended, the locally evolved solution returns to Σ with a value that is inconsistent with its initial data.

Flat cylinders with infinite extent in a timelike direction are obtained by identifying the left and right edges of the strip of Minkowski space between two parallel straight spacelike lines, $x = 0$ and $x = d$, after a time translation by τ :

$$(t, x = 0) \equiv (t + \tau, x = d). \quad (5)$$

For $\tau > d$, lines joining identified points are CTC's. These spacetimes are everywhere nonchronal: A CTC passes through every point, and there is no candidate for an initial value surface – no complete spacelike surface transverse to all timelike curves. On spacetimes with no generalized Cauchy surface, one can ask a related question, more closely tied to our knowledge of the universe's causal structure: whether the spacetime is *benign* [5]. A spacetime is benign if, at each point x , there is a finite spacelike surface S containing x for which arbitrary smooth data on S can be extended to a solution on the spacetime. For the massive wave equation, the cylinder spacetimes seem not even to be benign: A single massive particle leaving any spacelike surface S can be aimed to return to the surface at a point different from the one from which it left.

Problem Prove (or disprove) the conjecture that the flat cylinders, given by the identifications (5), are not benign for the massive scalar wave equation.

A spacetime M, g that avoids the problems so far encountered — the grandfather paradox and the lack of candidate Cauchy surface — is akin to spaces discussed by Geroch and Horowitz [9] and by Politzer [7]. Heuristically, as illustrated in Fig. 1, one removes from Minkowski space two parallel, timelike slits that are related by translation along a different timelike direction. The inner edges of the two slits are then glued; and the outer edges of the two slits are similarly glued. The formal construction and details of the initial value problem outlined below are given in Friedman and Morris [24]. Analysis of the initial value problem for a related spacetime with spacelike slits is given by Goldwirth et al. [10]

Because corresponding points on the left and right slits are related by a timelike translation, closed timelike curves extend from the left to the right slit, e.g., from the point labeled Q on the left to the identified point labeled $\mathcal{T}(Q)$ on the right. The *nonchronal* region A of a spacetime is the set of points through which there are closed timelike or null curves. Here it is a bounded region within the intersection of the past light cone of the top slit endpoint and the future light cone of the bottom slit endpoint. A hypersurface Σ that lies in the past of A and is a Cauchy surface of Minkowski space is an obvious candidate for a generalized Cauchy surface of M, g . In fact, it is easy to see that initial data in $L_2(\Sigma)$ leads to a solution in $L_2(M)$. In the past of A , solutions to the massless wave equation can be written as the sum $f(t - x) + g(t + x)$ of a right-moving and left-moving solution. To obtain a solution in the

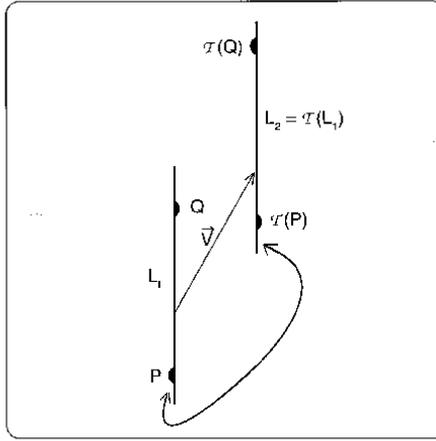


FIG. 1: A simple spacetime with CTCs and a generalized Cauchy surface is shown in this figure. Two parallel segments of equal length are removed from Minkowski space, two disjoint edges are joined to the left and right sides of each slit, and edge points related by the timelike translation \mathcal{T} are then identified.

spacetime M , one simply propagates left moving data that encounters the slit in the obvious way. For example, if a left-moving wave enters the left slit at Q , it emerges unaltered from the right slit at Q . The solution is unique. But it is discontinuous along future-directed null rays that extend from the endpoints of the slits, because the result of the wave propagation is to piece together solutions from disjoint parts of the initial data surface. For example, the right-going solutions on adjacent sides of right-directed null ray from any endpoint are Minkowski space solutions obtained from data that came from segments of Σ that are not adjacent.

The existence of solutions in L_2 , however, is not a generic property of two-dimensional spacetimes with a Cauchy horizon. If, for example, the two timelike slits were not parallel, the resulting spacetime would have an unstable Cauchy horizon: If Σ is a Cauchy surface for the past of A , data on Σ leads to a solution that diverges on the boundary of the past of A . The paradigm for this generic case is *Misner space* [6, 14, 15].

Misner space can be constructed by identifying the edges of a strip of Minkowski space between two parallel *null* lines. As in the previous example the CTCs of Misner space are confined to a bounded region (A is compact), and one can ask whether spacelike surfaces lying to the past or future of the nonchronal region A are generalized Cauchy surfaces.

To construct the space, let $u = t - x$, $v = t + x$, and consider the null strip $Bu_0 < u < u_0$, where $u_0 < 0$, and a boost \mathcal{B} of Minkowski space corresponding to velocity $V > 0$ is given by

$$u \rightarrow Bu, \quad v \rightarrow B^{-1}v, \quad (6)$$

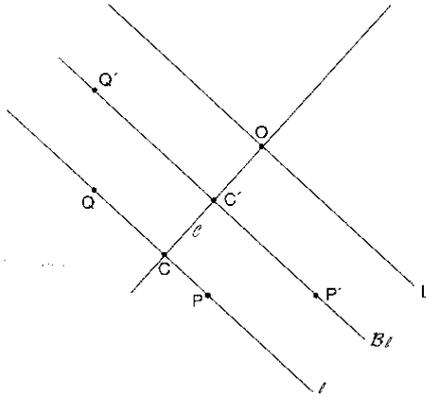


FIG. 2: Misner space is the region between the two null rays ℓ and $\mathcal{B}\ell$, with points of the null boundaries identified by the boost \mathcal{B} . The curve $\mathcal{C} = CC'$ is a chronology horizon, a closed null geodesic that separates the nonchronal region above it from the globally hyperbolic spacetime to its past.

with

$$B = \sqrt{\frac{1+V}{1-V}} \quad (7)$$

Points at the boundaries of the strip are identified after a boost:

$$(u_0, v) \equiv (Bu_0, B^{-1}v) \quad (8)$$

Identified points are spacelike separated for $v < 0$, null separated at $v = 0$ and timelike separated for $v > 0$. Closed timelike curves thus pass through each point of the region $v > 0$.

If ℓ is a null line parallel to L , and $\mathcal{B}\ell$ is its boosted image, then Misner space is the strip between ℓ and $\mathcal{B}\ell$, where boundary points related by \mathcal{B} are identified (see Fig. 2). Misner space has a single closed null geodesic, $\mathcal{C} = CC'$ in the Figure, and the past \mathcal{P} of \mathcal{C} is globally hyperbolic. The future of \mathcal{C} is nonchronal, so \mathcal{C} is a chronology horizon, a Cauchy horizon that bounds the nonchronal region. Initial data for the scalar-wave equation can be posed on a Cauchy surface Σ of \mathcal{P} , but solutions have divergent energy on the chronology horizon.

Note that this globally hyperbolic past part of Misner space can be obtained from 1-dimensional Minkowski spacetime whose walls are moving toward each other – by identifying left and right walls at the same proper time read by clocks on each wall (see, e.g., Thorne [1994]). This construction makes it obvious that light rays are boosted each time they traverse the space, in the same way that a light ray is boosted when reflected by a moving mirror.

The reason solutions diverge is then clear in the geometrical optics limit. A light ray γ , starting from Σ , loops about the space and is boosted each time it loops.[?]] Because γ loops an infinite number of times before reaching \mathcal{C} , its frequency and energy diverge as it approaches the horizon. The ray γ is an incomplete geodesic: It reaches the horizon in finite affine parameter length, because each boost decreases the affine parameter by the blueshift factor $[(1+V)/(1-V)]^{1/2}$, with the velocity of the boost V .

This behavior is not unique to Misner space: A theorem due to Tipler [16] shows that geodesic incompleteness is generic in spacetimes like Misner space in which CTCs are “created” – spacetimes with a nonchronal region to the future of a spacelike hypersurface. And a similar argument by Hawking underpins the classical part of his Chronology Protection Conjecture. [17] (See also Chrusciel and Isenberg [18], who show that the generic, compactly generated horizon has generators whose structure is more complex than that considered by Hawking.)

When the horizon is not compactly generated, classical fields need not diverge, and a class of Gott spacetimes [8] serve as an example. Cutler [11] shows that a spacelike hypersurface Σ extends to spatial infinity and lies to the past of the nonchronal region. Here CTCs run to spatial infinity. These characteristics hold for the particular Gott spacetime introduced here, but an additional key feature is that its covering space is three-dimensional Minkowski space (with images of the string singularities removed). Carinhas [12] has shown for the massless scalar wave equation that data on Σ satisfying suitable asymptotic conditions leads to solutions on a set of Gott spacetimes (see also Boulware [13]).

IV. EXISTENCE AND UNIQUENESS THEOREMS FOR SOME FOUR DIMENSIONAL SPACETIMES WITH CTCs

In four spacetime dimensions, existence and uniqueness theorems have been proved for a class of stationary, asymptotically flat spacetimes.[19, 24] In these spacetimes, the nonchronal region is bounded in space, but there is no Cauchy horizon and CTCs are always present. Because the spacetimes are asymptotically flat one can define \mathcal{I} . In Minkowski space \mathcal{S}^- is a generalized Cauchy surface for massless wave equations, and the goal here is to show that \mathcal{S}^- is also a generalized Cauchy surface for a class of spacetimes with CTCs.

We first consider spacetimes with topology $\mathcal{N} = \mathcal{M} \times \mathbb{R}$, where \mathcal{M} is a plane with a handle (wormhole) attached: $\mathcal{M} = \mathbb{R}^3 \# (S^2 \times S^1)$. The metric $g_{\alpha\beta}$ on \mathcal{N} is smooth (C^∞), and, for simplicity in treating the asymptotic behavior of the fields, we will assume that outside a compact region \mathcal{R} the geometry is flat, with metric $\eta_{\alpha\beta}$.

One can construct the 3-manifold \mathcal{M} from \mathbb{R}^3 by removing two balls and identifying their spherical boundaries, Σ_I and Σ_{II} , as shown in Fig. 3. The sphere obtained by the identification will be called

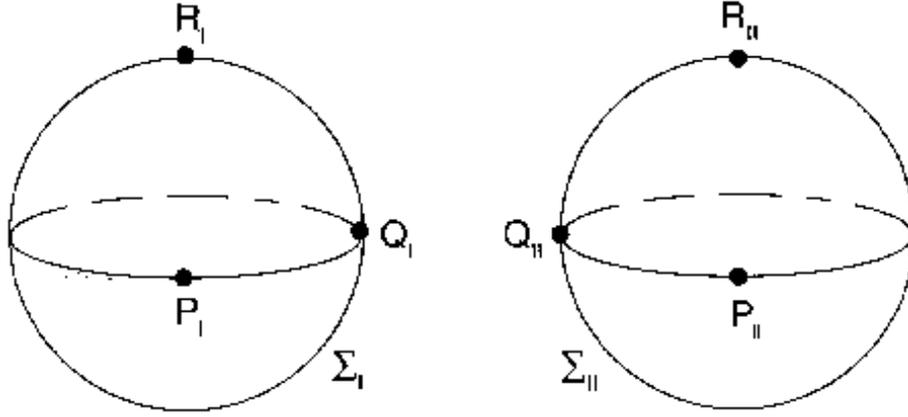


FIG. 3: An orientable 3-manifold M is constructed by identifying points of Σ_I and points of Σ_{II} that are labeled by the same letter, with subscripts I and II.

the “throat” of the handle. (Its location is arbitrary: After removing any sphere, Σ , from the handle of \mathcal{M} one is left with a manifold homeomorphic to $\mathbb{R}^3 \setminus (B^3 \# B^3)$, whose boundary is the disjoint union of two spheres.) One can similarly construct the spacetime \mathcal{N} from \mathbb{R}^4 by removing two solid cylinders and identifying their boundaries C_I and C_{II} . We will denote by \mathcal{T} the map from C_I to C_{II} that relates identified points. For the spacetimes we will consider, the identified points will be timelike separated.

A static metric on \mathcal{N} is given by

$$g_{\alpha\beta} = -e^{-2\nu} t_\alpha t_\beta + h_{\alpha\beta}, \quad (9)$$

where $h_{\alpha\beta} t^\beta = 0$.

If the Minkowski coordinate t is extended to $\mathcal{N} \setminus C$ by making \mathcal{M}_t a t -constant surface, then $t^\alpha \nabla_\alpha t = 1$, $\nabla^\alpha t = -e^{-2\nu} t^\alpha$, and the metric (9) can be written on $\mathcal{N} \setminus C$ in the form

$$g_{\alpha\beta} = -e^{2\nu} \partial_\alpha t \partial_\beta t + h_{\alpha\beta}. \quad (10)$$

It will be convenient to single out a representative hypersurface,

$$\mathcal{M} := \mathcal{M}_0. \quad (11)$$

We will denote by h_{ab} the corresponding spatial metric on \mathcal{M} ; that is, h_{ab} is the pullback of $h_{\alpha\beta}$ (or $g_{\alpha\beta}$) to \mathcal{M} .

We consider the wave equation

$$\square \Phi \equiv \nabla^\alpha \nabla_\alpha \Phi = 0, \quad (12)$$

for a massless scalar field Φ .

Initial data in Minkowski space on a future null cone is simply a specification of the field Φ on that cone. On \mathcal{I}^- , the field itself vanishes, but the field rescaled by a radial coordinate is finite; initial data on \mathcal{I}^- can then be written in terms of the standard ingoing null coordinate v , radial coordinate r , and unit radial vector $\hat{\mathbf{r}}$ as

$$f(v, \hat{\mathbf{r}}) = \lim_{r \rightarrow \infty} r\Phi(v, r\hat{\mathbf{r}}). \quad (13)$$

The same data on \mathcal{I}^- determines a solution to the wave equation in the spacetimes \mathcal{N}, g .

Proposition 1. *For almost all spacetimes \mathcal{N}, g of the kind just described (for almost all parameters τ), the following existence theorem holds. Let f be initial data on \mathcal{I}^- for which f and all its derivatives are in $L_2(\mathcal{I}^-)$. Then there exists a solution Φ to the scalar wave equation that is smooth and asymptotically regular at null and spatial infinity and which has f as initial data.*

Because the geometry is static, we can express solutions as a superposition of functions with harmonic time dependence. The fact that there is no foliation by spacelike slices leads to a lack of orthogonality of the eigenfunctions, and the spectral theorem cannot be used. Instead we explicitly prove convergence of a superposition of the form

$$\Phi(t, x) = \int d\omega \phi(\omega, x) e^{-i\omega t}. \quad (14)$$

Here x is naturally a point of the manifold of trajectories of t^α , but we can identify it with a point of a simply connected spacelike hypersurface \mathcal{M} , with spherical boundaries Σ_I and Σ_{II} . Let (t, x_I) and $(t + \tau, x_{II})$ be points of $\mathcal{N} \setminus \mathcal{C}$ that are identified in \mathcal{N} . Continuity of Φ and its normal derivative at the identified points is expressed by

$$\Phi(x_{II}) = \Phi(x_I) \quad (15)$$

$$\hat{n}_{II} \cdot \nabla \Phi(x_{II}) = -\hat{n}_I \cdot \nabla \Phi(x_I). \quad (16)$$

The harmonic components of Φ on \mathcal{N} can be regarded as fields $\phi(\omega, x)$ on \mathcal{M} satisfying elliptic equations of the form

$$(\omega^2 + \mathcal{L})\phi = 0, \quad (17)$$

where \mathcal{L} can be defined by the action of $\nabla_\alpha \nabla^\alpha$ on time independent fields f on \mathcal{N} :

$$\mathcal{L} = e^\nu D^\alpha e^\nu D_\alpha, \quad (18)$$

and D_a is the covariant derivative of the 3-metric h_{ab} on \mathcal{M} . The major difficulty lies in the fact that, because the boundary conditions (15,16) involve a time-translation by Killing parameter τ , the

corresponding boundary conditions on the harmonic components ϕ depend on the frequency ω , via a phase $\eta = \omega\tau$:

$$\phi(\omega, x_{II}) = e^{i\eta}\phi(\omega, x_I), \quad (19)$$

$$\hat{n}_{II} \cdot \nabla\phi(\omega, x_{II}) = -e^{i\eta}\hat{n}_I \cdot \nabla\phi(\omega, x_I). \quad (20)$$

As a result, eigenfunctions associated with different frequencies are eigenfunctions of different operators; they are not orthogonal, and their completeness is not guaranteed by the spectral theorem.

Instead, the following steps outline the construction of a solution.

1. For a fixed value η of the phase, *the operator \mathcal{L}_η with boundary conditions (19,20) is self-adjoint on the space $L_2(\mathcal{M})$ with domain \mathbb{H}_2 .*

Here the boundary conditions enforce the symmetry of the operator

$$\langle f | \mathcal{L}_\eta g \rangle = \langle \mathcal{L}_\eta f | g \rangle = \langle h | g \rangle, \quad (21)$$

by requiring that the current entering Σ_I coincide with the current leaving Σ_{II} :

$$\int_{\Sigma_{II}} dS_a e^{-\nu} (\bar{f} D^a g - \bar{g} D^a f) + \int_{\Sigma_I} dS_a e^{-\nu} (\bar{f} D^a g - \bar{g} D^a f). \quad (22)$$

2. Eigenfunctions exist whose incoming part coincides with the incoming part of a plane wave for each wavevector k . These are solutions $F(\eta, k, x)$ to Eq. (17) that, for $r > R$, have the form

$$F = (2\pi)^{-3/2} e^{ik \cdot x} + \text{outgoing waves.}, \quad (23)$$

Existence is proved, following Wilcox [25], by the limiting absorption method: One adds an imaginary part to the frequency $\omega = |k|$. Because \mathcal{L}_η is self-adjoint, $\mathcal{L}_\eta + i\epsilon$ is invertible in L_2 . One can rewrite the homogeneous equation $\mathcal{L}_{\eta+i\epsilon} F = 0$, for F with asymptotic behavior (23), as an inhomogeneous equation $\mathcal{L}_{\eta+i\epsilon} F_{\text{out}} = \rho$, with F_{out} purely outgoing for $r > R$. The sign of the imaginary part of the frequency enforces an outgoing solution, and F is then found from the limit, as the imaginary part goes to zero, of a family of functions F_{out} in L_2 .

3. In flat space, the solution for data f on \mathcal{I}^- can be written in terms of the fourier transform \tilde{f} of f in the form

$$\Phi(t, x) = \text{Re } 2 \int d^3k a(k) e^{i(k \cdot x - \omega t)},$$

where

$$\tilde{f}(\omega, \hat{r}) = i\omega a(-\omega \hat{r}), \quad \omega \geq 0.$$

Here, $e^{ik \cdot x}$ is replaced by $F(\eta = \omega\tau, k, x)$, and one shows convergence of their superposition,

$$\Phi(t, x) = \operatorname{Re} 2 \int dk F(\eta = \omega\tau, k, x) a(k). \quad (24)$$

Although one cannot directly use the spectral theorem, convergence of the integral does rely on a related unitarity relation for the eigenfunctions F of the Hermitian operator \mathcal{L}_η for fixed boundary phase η . That is, regarded as a map from $L_2(\mathcal{M})$ to $L_2(\mathbb{R}^3)$, F is norm-preserving:

$$\|g\|_{L_2(\mathcal{M})} = \left\| \int dV_x F(\eta, k, x) g(x) \right\|_{L_2(\mathbb{R}^3)} \quad (25)$$

This allows us to bound the norm of a truncated fourier transform of F : Let χ be a smooth step function, satisfying

$$\chi(x) = \begin{cases} 0, & r > R + \epsilon \\ 1, & r < R \end{cases}$$

$$\hat{F}(\eta, k, y) := \int dV_x e^{-\nu} F(\eta, k, x) \chi(x) \quad (26)$$

has uniformly bounded norm in k -space,

$$\left\| \hat{F}(\eta, \cdot, y) \right\|_{L_2(\mathbb{R}^3)} \leq CR^{3/2}, \quad \forall \eta, y. \quad (27)$$

From this uniform bound, one can show

$$\int d\tau dk dy \frac{\omega^{2n} |\hat{F}(\omega\tau, k, y)|^2}{(1 + \omega^2)^n (1 + y^2)^{3/2 + \epsilon}} < \infty.$$

This inequality, in turn implies

$$\begin{aligned} \omega^n \hat{F}(\omega\tau, k, y) &\in L_2(I) \otimes L_{2, -n}(\mathbb{R}^3) \otimes L_{-3/2 - \epsilon}(\mathbb{R}^3) \\ \implies \nabla^n F(\omega\tau, k, x) &\in L_2(I) \otimes L_{2, -n}(\mathbb{R}^3) \otimes H_{-3/2 - \epsilon}(\mathcal{M}_D) \\ \implies F(\omega\tau, k, x) &\in L_2(I) \otimes L_{2, -n}(\mathbb{R}^3) \otimes H_{n-3/2 - \epsilon}(\mathcal{M}_D). \end{aligned} \quad (29)$$

Thus, for almost all τ ,

$$\begin{aligned} F(\omega\tau, k, x) &\in L_{2, -n}(\mathbb{R}^3) \otimes H_{n-3/2 - \epsilon}(\mathcal{M}_D) \\ \implies \int dk a(k) F(\omega\tau, k, x) &\in H_{n-3/2 - \epsilon}(\mathcal{M}_D) \end{aligned} \quad (30)$$

Finally, $f \in H_n(\mathcal{S}^-)$, all n , implies $\hat{f} \in L_{2, -n}$, all n , whence Φ given by Eq. (24) is smooth.

4. Asymptotic regularity of Φ follows from its explicit form for $r > R$ in terms of the value of Φ and $\nabla\Phi$ at $r = R$. That is, one can use the flat-space Green function to write Φ outside $r = R$.

More recently, Bachelot [19] has proved a similar existence theorem and a strong uniqueness theorem for another family of stationary, four-dimensional spacetimes that are flat outside a spatially compact

region. These spacetimes have Euclidean topology and their nonchronal regions have topology (solid torus) $\times \mathbb{R}$. The metric is axisymmetric, with one free function a that describes the tipping of the light cones in the direction of the rotational Killing vector $\partial\phi$.

$$g = -(dt - a d\phi)^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2 d\phi^2. \quad (31)$$

Circles about the axis of symmetry are CTCs when ∂_ϕ is spacelike — that is, when $r \sin \theta < a$. By choosing $a = 0$ outside a torus, one can restrict CTCs to the interior of a smaller torus. Again data at \mathcal{I}^- for Φ yields a smooth, asymptotically regular solution Φ , and Bachelot shows that Φ is unique. This is a significantly stronger result than the weak uniqueness obtained for the wormhole spacetimes described above, and it suggests that a strong uniqueness theorem should hold for those spacetimes as well.

V. CONJECTURES FOR MORE GENERAL FOUR-DIMENSIONAL SPACETIMES

As noted in Sect. III four-dimensional spacetimes that have Cauchy horizons and satisfy the null energy condition are geodesically incomplete. In two dimensions, an incomplete null geodesic that approaches a closed null geodesic as it approaches the chronology horizon leads to instability of that horizon. In four dimensions, however, an incomplete null geodesic γ does not always imply that the chronology horizon is unstable. This is because there may be only a set of measure zero of such geodesics, so that the energy may remain finite on the chronology horizon. For the time-tunnel spacetimes considered in refs [20–22], a congruence of null rays initially parallel to γ spreads as the rays approach the chronology horizon. When the spreading of the rays overcomes the successive boosts (when the fractional decrease in flux is greater than the fractional increase in squared frequency), the horizon is stable in the geometrical optics approximation, and we will call it *optically stable*. (A precise, but long-winded definition of optical stability is given in Ref. [24];

a similar definition, applicable in a more restricted context, is given by Hawking[17]). Because the instability of the chronology horizon (or of the spacetime to its future) appears to be the obstacle to existence of solutions for data on candidate generalized Cauchy surfaces, we are led to a conjecture that relates optical stability to the existence of solutions.

Existence Conjecture. Let \mathcal{N}, g be a smooth, asymptotically flat spacetime for which past and future regions $\mathcal{P} = \mathcal{N} \setminus J^+(A)$ and $\mathcal{F} = \mathcal{N} \setminus J^-(A)$ of a compact 4-dimensional submanifold A are globally hyperbolic. If \mathcal{N}, g is optically stable, solutions to massless wave equations (for scalar, Maxwell, and Weyl fields) exist on \mathcal{N}, g for smooth data on a Cauchy surface for \mathcal{P} .

A conjecture relating uniqueness for massless fields to uniqueness in a geometric-optics sense is easier to formulate.

Uniqueness Conjecture. Again let \mathcal{N}, g be a smooth, asymptotically flat spacetime for which past and future regions $\mathcal{P} = \mathcal{N} \setminus J^+(A)$ and $\mathcal{F} = \mathcal{N} \setminus J^-(A)$ of a compact 4-dimensional submanifold A are globally hyperbolic. Let S_{\pm} be Cauchy surfaces for $\mathcal{N} \setminus J^{\pm}(A)$. If all but a set of measure zero of null geodesics intersect S_+ and S_- , then solutions to massless wave equations on \mathcal{N} are unique for initial data on S_- (and for initial data on S_+).

If one omits the restriction on null geodesics, uniqueness fails: It is not difficult to construct spacetimes satisfying the remaining conditions of the conjecture for which solutions to the massless scalar wave equation have support on a compact region.[24] One example begins with a 4-torus with flat Lorentz metric chosen to make two of the generators null and the other two spacelike. The metric allows a nonzero plane-wave solution whose support is not the entire torus. One can smoothly glue the torus to an asymptotically flat Lorentian spacetime without altering the metric on the support of the scalar field.

For no asymptotically flat spacetime in 4-dimensions, in which CTCs are confined to a compact region, am I aware of a rigorous demonstration that finite-energy solutions to the scalar wave equation do exist for arbitrary initial data, or that solutions are unique.

Still less is known about interacting fields.

The well-known billiard-ball examples of Echeverria *et al.*[22, 26] are the basis for our present intuitive understanding. These examples exhibit a multiplicity of solutions for the same initial data, suggesting that uniqueness in spacetimes with CTCs is likely to hold only for free or weakly interacting fields. Because solutions seem always to exist for the billiard ball examples in the spacetimes they considered, it may be that classical interacting fields have solutions on spacetimes for which solutions to the free field equations exist.

Fewster, Higuchi and Wells [27] looked at a model of an interacting field theory in which space is discrete, and time is identified to obtain a discrete version of 2-dimensional Minkowski space with two horizontal slits removed and opposite edges of the slits identified. The field ψ satisfies an equation of the form

$$\partial_t \psi = L\psi + \lambda \psi^\dagger \psi \psi,$$

where L is a linear operator, and λ is real.

Fewster *et al.* find that solutions exist for arbitrary data and arbitrary λ and that they are unique for small λ . For large λ , however, uniqueness is lost.

An obvious question is whether generalized Cauchy surfaces for free fields similarly serve as generalized Cauchy surfaces for weakly interacting fields; and whether, as the toy models suggest, uniqueness fails

above some critical value of the interaction parameter.

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- [] That is, trajectories of a (locally-defined) timelike Killing vector cross the null geodesic at a sequence of points. The Killing vector can be used to compare the affine parameter at successive crossing points by time-translating a segment of the geodesic to successively later segments. Compared in this way, the affine parameter of a given segment will be less than that of the next segment by the blueshift factor $[(1 + V)/(1 - V)]^{1/2}$.