CHEEGER-GROMOV THEORY AND APPLICATIONS TO GENERAL RELATIVITY

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Contents

This paper surveys aspects of the convergence and degeneration of Riemannian metrics on a given manifold M, and some recent applications of this theory to general relativity. The basic point of view of convergence/degeneration described here originates in the work of Gromov, cf. [27]-[29], with important prior work of Cheeger [14], leading to the joint work of [16].

This Cheeger-Gromov theory assumes L^∞ bounds on the full curvature tensor. For reasons discussed below, we focus mainly on the generalizations of this theory to spaces with L^∞, (or L^p) bounds on the Ricci curvature. Although versions of the results described hold in any dimension, for the most part we restrict the discussion to 3 and 4 dimensions, where stronger results hold and the applications to general relativity are most direct.

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1. Background: Examples and Definitions.

The space $\mathbb{M}$ of Riemannian metrics on a given manifold M is an infinite dimensional cone, (in the vector space of symmetric bilinear forms on M), and so is highly non-compact. Arbitrary sequences of Riemannian metrics can degenerate in very complicated ways.

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On the other hand, there are two rather trivial but nevertheless important sources of non-compactness.

• *Diffeomorphisms.* The group $\mathcal{D}$ of diffeomorphisms of M is non-compact and acts properly on $\mathbb{M}$ by pullback. Hence, if g is any metric in $\mathbb{M}$ and ϕ_i is any divergent sequence of diffeomorphisms, then $g_i = \phi_i^* g$ is a divergent sequence in $\mathbb{M}$. However, all the metrics g_i are isometric, and so are indistinguishable metrically. In terms of a local coordinate representation, the metrics g_i locally are just different representatives of the fixed metric g.

Thus, for most problems, one considers only equivalence classes of metrics $[g]$ in the moduli space

$$\mathcal{M} = \mathbb{M}/\mathcal{D}.$$

(A notable exception is the Yamabe problem, which is not well-defined on $\mathcal{M}$).

• *Scaling.* For a given metric g and parameter $\lambda > 0$, let $g_\lambda = \lambda^2 g$ so that all distances are rescaled by a factor of λ. If $\lambda \to \infty$, or $\lambda \to 0$, the metrics g_λ diverge. In the former case, the manifold (M, g_λ), say compact, becomes arbitrarily large, in that global invariants such as diameter, volume, etc. diverge to infinity; there is obviously no limit metric. In the latter case, (M, g_λ) converges, as a family of metric spaces, to a single point. Again, there is no limiting Riemannian metric on M.

Although one has divergence in both cases described above, they can be combined in natural ways to obtain convergence. Thus, for g_λ as above, suppose $\lambda \to \infty$, and choose any fixed point $p \in M$. For any fixed $k > 0$, consider the geodesic ball $B_p = B_p(k/\lambda)$, so the g-radius of this ball is $k/\lambda \to 0$, as $\lambda \to \infty$. On the other hand, in the metric g_λ, the ball B_p is a geodesic ball of fixed radius k. Since k/λ is small, one may choose a local coordinate system $\mathcal{U} = \{u_i\}$ for B_p, with p mapped to the origin in $\mathbb{R}^n$. Let $u_i^\lambda = \lambda u_i = \phi_\lambda \circ u_i$, where $\phi_\lambda(x) = \lambda x$. Thus ϕ_λ is a divergent sequence of diffeomorphisms of $\mathbb{R}^n$, and $\mathcal{U}_\lambda = \{u_i^\lambda\}$ is a new collection of charts. One then easily sees that

$$g_\lambda(\partial/\partial u_i^\lambda, \partial/\partial u_j^\lambda) = g(\partial/\partial u_i, \partial/\partial u_j) = g_{ij}. \tag{1.1}$$

As $\lambda \to \infty$, the ball B_p shrinks to the point p and the coefficients g_{ij} tend to the constants $g_{ij}(p)$. On the other hand, the metrics g_λ are defined on the intrinsic geodesic ball of radius k. Since k is arbitrary, the metrics $\phi_\lambda^* g_\lambda$ converge smoothly to the limit flat metric g_0 on the tangent space $T_p(M)$, induced by the inner product g_p on $T_p(M)$,

$$(M, \phi_\lambda^* g_\lambda) \to (T_p M, g_0). \tag{1.2}$$

This process is called "blowing up", since one restricts attention to smaller and smaller balls, and blows them up to a definite size. Note that the part of M at any definite g-distance to p escapes to infinity, and is not detected in the limit g_0. Thus, it is important to attach base points to the blow-up construction; different base points may give rise to different limits, (although in this situation all pointed limits are isometric).

There is an analogous, although more subtle blowing up process for Lorentzian metrics due to Penrose, where the limits are non-flat plane gravitational waves, cf. [37].

If (M, g) is complete and non-compact, one can carry out a similar procedure with $\lambda \to 0$, called "blowing down", where geodesic balls, (about a given point), of large radius $B_p(k/\lambda)$ are rescaled down to unit size, i.e. size k. This is of importance in understanding the large scale or asymptotic behavior of the metric and will arise in later sections.

This discussion leads to the following definition for convergence of metrics. Let Ω be a domain in $\mathbb{R}^n$ and let $C^{k,\alpha}$ denote the usual Hölder space of C^k functions on Ω with α-Hölder continuous k^{th} partial derivatives. Similarly, let $L^{k,p}$ denote the Sobolev space of functions with k weak derivatives in L^p. Since one works only locally, we are only interested in the local spaces $C^{k,\alpha}_{loc}$ and $L^{k,p}_{loc}$ and corresponding local norms and topology.

Definition 1.1. A sequence of metrics g_i on n-manifolds M_i is said to converge in the $L^{k,p}$ topology to a limit metric g on the n-manifold M if there is a locally finite collection of charts $\{\phi_k\}$ covering M, and a sequence of diffeomorphisms $F_i : M \to M_i$, such that

$$(F_i^* g_i)_{\alpha\beta} \to g_{\alpha\beta}, \tag{1.3}$$

in the $L^{k,p}_{loc}$ topology. Here $(F_i^* g_i)_{\alpha\beta}$ and $g_{\alpha\beta}$ are the local component functions of the metrics $F_i^* g_i$ and g in the charts ϕ_k.

The same definition holds for convergence in the $C^{k,\alpha}$ topology, as well as the weak $L^{k,p}$ topology. (Recall that a sequence of functions $f_i \in L^p(\Omega)$ converges weakly in L^p to a limit $f \in L^p(\Omega)$ iff $\int f_i g \to \int fg$, for all $g \in L^q(\Omega)$, where $p^{-1} + q^{-1} = 1$).

It is easily seen that this definition of convergence is independent of the choice of charts $\{\phi_k\}$ covering M. The manifolds M and M_i are not required to be closed.

In order to obtain local control on a metric, or sequence of metrics, one assumes curvature bounds. The theory described by Cheeger-Gromov requires a bound on the full Riemann curvature tensor

$$|Riem| \leq K, \tag{1.4}$$

for some $K < \infty$. Since the number of components of the Riemann curvature is much larger than that of the metric tensor itself, (in dimensions ≥ 4), this corresponds to an overdetermined set of constraints on the metric and so is overly restrictive. It is much more natural to impose bounds on the Ricci curvature

$$|Ric| \leq k, \tag{1.5}$$

since the Ricci curvature is a symmetric bilinear form, just as the metric is. Of course, assuming bounds on Ricci is natural in general relativity, via the Einstein equations. Thus throughout the paper, we emphasize (1.5) over (1.4) whenever possible.

The Cheeger-Gromov theory may be viewed as a vast generalization of the basic features of Teichmüller theory to higher dimensions and variable curvature, (although it was not originally phrased in this way). Recall that Teichmüller theory describes the moduli space $\mathcal{M}_c$ of constant curvature metrics on surfaces. On closed surfaces, one has a *basic trichotomy* for the behavior of sequences of such metrics, normalized to unit area:

• *Compactness/Convergence.* A sequence $g_i \in \mathcal{M}_c$ has a subsequence converging smoothly, (C^∞), to a limit metric $g \in \mathcal{M}_c$. As in the definition above, the convergence is understood to be modulo diffeomorphisms. For instance this is always the case on S^2, since the moduli space $\mathcal{M}_c$ is a point for S^2.

• *Collapse.* The sequence $g_i \in \mathcal{M}_c$ collapses everywhere, in that

$$inj_{g_i}(x) \to 0, \tag{1.6}$$

at every x, where inj_{g_i} is the injectivity radius w.r.t. g_i. This collapse occurs only on the torus T^2 and such metrics become very long and very thin. There is no limit metric on T^2. Instead, by choosing (arbitrary) base points x_i, one may consider based sequences (T^2, g_i, x_i), whose limits are then the collapsed space $(\mathbb{R}, g_\infty, x_\infty)$.

• *Cusp Formation.* This is a mixture of the two previous cases, and occurs only for hyperbolic metrics, i.e. on surfaces Σ_g of genus $g \geq 2$. In this case, there are based sequences (Σ_g, g_i, x_i) which converge to a limit $(\Sigma, g_\infty, x_\infty)$ which is a complete non-compact hyperbolic surface of finite volume, hence with a finite number of cusp ends $S^1 \times \mathbb{R}^+$. The convergence is smooth, and uniform on compact subsets. As one goes to infinity in any cusp end $S^1 \times \mathbb{R}^+$, the limit metric collapses in the sense that $inj_{g_\infty}(z_k) \to 0$, as $z_k \to \infty$. There are other based sequences (Σ, g_i, y_i) which collapse, i.e. (1.6) holds on domains of arbitrarily large but bounded diameter about y_i. As before, limits of such sequences are of the form $(\mathbb{R}, g_\infty, y_\infty)$.

2. Convergence/Compactness.

To prove the (pre)-compactness of a family of metrics, or the convergence of a sequence of metrics, the main point is to establish a lower bound on the radius of balls on which one has apriori control of the metric in a given topology, say $C^{k,\alpha}$ or $L^{k,p}$. Given such uniform local control, it is then usually straightforward to obtain global control, via suitable global assumptions on the volume or diameter. (Alternately, one may work instead on domains of bounded diameter).

To obtain such local control, the first issue is to choose a good "gauge", i.e. representation of the metric in local coordinates. For this, it is natural to look at coordinates built from the geometry of the metric itself. In the early stages of development of the theory, geodesic normal coordinates were used. Later, Gromov [27] used suitable distance coordinates. However, both these coordinate systems entail loss of derivatives - two in the former case, one in the latter. It is now well-known that Riemannian metrics have optimal regularity properties in harmonic coordinates, cf. [21]; this is due to the special form of the Ricci curvature in harmonic coordinates, known to relativists long ago.

Given the choice of harmonic gauge, it is natural to associate a harmonic radius $r_h : M \to \mathbb{R}^+$, which measures the size of balls on which one has harmonic coordinates in which the metric is well controlled. The precise definition, cf. [1], is as follows.

Definition 2.1. Fix a function topology, say $L^{k,p}$, and a constant $c_o > 1$. Given $x \in (M, g)$, define the $L^{k,p}$ harmonic radius to be the largest radius $r_h(x) = r_h^{k,p}(x)$ such that on the ball $B_x(r_h(x))$ one has a harmonic coordinate chart $U = \{u_\alpha\}$ in which the metric $g = g_{\alpha\beta}$ is controlled in $L^{k,p}$ norm: thus,

$$c_o^{-1}\delta_{\alpha\beta} \le g_{\alpha\beta} \le c_o\delta_{\alpha\beta}, \quad \text{(as bilinear forms)}, \tag{2.1}$$

$$[r_h(x)]^{kp-n} \int_{B_x(r_h(x))} |\partial^k g_{\alpha\beta}|^p dV \le c_o - 1. \tag{2.2}$$

Here, it always assumed that $kp > n = dim M$, so that $L^{k,p}$ embeds in C^0, via Sobolev embedding. The precise value of c_o is usually unimportant, but is understood to be fixed once and for all. Both estimates in (2.1)-(2.2) are scale invariant, (when the harmonic coordinates are rescaled as in (1.1)), and hence the harmonic radius scales as a distance.

Note that if $r_h(x)$ is large, then the metric is close to the flat metric on large balls about x, while if $r_h(x)$ is small, then the derivatives of $g_{\alpha\beta}$ up to order k are large in L^p on small balls about x. Thus, the harmonic radius serves as a measure of the degree of concentration of $g_{\alpha\beta}$ in the $L^{k,p}$ norm.

It is important to observe that the harmonic radius is continuous with respect to the (strong) $L^{k,p}$ topology on the space of metrics, cf. [1], [3]. In general, it is not continuous in the weak $L^{k,p}$ topology.

One may define such harmonic radii w.r.t. other topologies, for instance $C^{k,\alpha}$ in a completely analogous way; these have the same properties.

Suppose g_k is a sequence of metrics on a manifold M, (possibly open), with a uniform lower bound on r_h. On each ball, one then has $L^{k,p}$ control of the metric components. The well-known Banach-Alaoglu theorem, (bounded sequences are weakly compact in Banach spaces), then implies that the metrics on the ball have a weakly convergent subsequence in $L^{k,p}$, so one obtains a limit metric on each ball. It is straightforward to verify that the overlaps of these charts are in $L^{k+1,p}$, and so one has a limit $L^{k,p}$ metric on M. The convergence to limit is in the weak $L^{k,p}$ topology and uniform on compact subsets. Strictly speaking, one also has to prove that the harmonic coordinate charts for g_k also converge, or more precisely may be replaced by a fixed coordinate chart, but this also is not difficult, cf. [1], [3] for details.

The same type of arguments hold w.r.t. the $C^{k,\alpha}$ topology, via the Arzela-Ascoli theorem; here weak $L^{k,p}$ convergence is replaced by convergence in the $C^{k,\alpha'}$ topology, for $\alpha' < \alpha$.

Thus, the main issue in obtaining a convergence result is to obtain a lower bound on a suitable harmonic radius r_h under geometric bounds. The following result is one typical example.

Theorem 2.2. (Convergence I). *Let M be a closed n-manifold and let $\mathcal{M}(\lambda, i_o, D)$ be the space of Riemannian metrics such that*

$$|Ric| \leq k, \ inj \geq i_o, \ diam \leq D. \tag{2.3}$$

Then $\mathcal{M}(\lambda, i_o, D)$ is precompact in the $C^{1,\alpha}$ and weak $L^{2,p}$ topologies, for any $\alpha < 1$ and $p < \infty$.

Thus, for any sequence, there is a subsequence which converges, in these topologies, to a limit $C^{1,\alpha} \cap L^{2,p}$ metric g_∞ on M.

Sketch of Proof: As discussed above, it suffices to prove a uniform lower bound on the $L^{2,p}$ harmonic radius $r_h = r_h^{2,p}$, i.e.

$$r_h(x) \geq r_o = r_o(k, i_o, D), \tag{2.4}$$

under the bounds (2.3).

Overall, the proof of (2.4) is by contradiction. Thus, if (2.4) is false, there is a sequence of metrics g_i on M, satisfying the bounds (2.3), but for which $r_h(x_i) \to 0$, for some points $x_i \in M$. Without loss of generality, (since M is closed), assume that the base points x_i realize the minimal value of r_h on (M, g_i). Then rescale the metrics g_i by this minimal harmonic radius, i.e. set

$$\bar{g}_i = r_h(x_i)^{-2} \cdot g_i. \tag{2.5}$$

If $\bar{r}_h$ denotes the harmonic radius w.r.t. $\bar{g}$, by scaling properties one has

$$\bar{r}_h(x_i) = 1, \quad \text{and} \quad \bar{r}_h(y_i) \geq 1, \tag{2.6}$$

for all $y_i \in (M, \bar{g}_i)$. By the remarks preceeding the proof, the pointed Riemannian manifolds $(M, \bar{g}_i, x_i)$ have a subsequence converging in the *weak* $L^{2,p}$ topology to a limit $L^{2,p}$ Riemannian manifold $(N, \bar{g}_\infty, x_\infty)$. (Again, this convergence is understood to be modulo diffeomorphisms, as in Definition 1.1). Of course $diam_{\bar{g}_i} M \to \infty$, so that the complete open manifold N is distinct from the original compact manifold M. The convergence is uniform on compact subsets.

So far, nothing essential has been done - the construction above more or less amounts to just renormalizations. There are two basic ingredients in obtaining further control however, one geometric and one analytic.

We begin with the geometric argument. The limit space $(N, \bar{g}_\infty)$ is Ricci-flat, since the bound (2.3) on the Ricci curvature of g_i becomes in the scale $\bar{g}_i$,

$$|Ric_{\bar{g}_i}| \leq k \cdot r_h(x_i) \to 0, \quad \text{as} \quad i \to \infty. \tag{2.7}$$

Actually, it is Ricci-flat in a weak sense, since the convergence is only in weak $L^{2,p}$. However, it is easy to see, (cf. also below), that weak $L^{2,p}$ solutions of the (Riemannian) Einstein equations are real-analytic, and so the limit is in fact a smooth Ricci-flat metric.

Next, by (2.3), the injectivity radius of $\bar{g}_i$ satisfies

$$inj_{\bar{g}_i} \geq i_o \cdot r_h(x_i)^{-1} \to \infty, \quad \text{as} \quad i \to \infty, \tag{2.8}$$

so that, roughly speaking, the limit $(N, \bar{g}_\infty)$ has infinite injectivity radius at every point. More importantly, the bound (2.8) implies that $(M, \bar{g}_i)$ contains arbitrarily long, (depending on i), minimizing geodesics in any given direction through the center point x_i. It follows that the limit $(N, \bar{g}_\infty)$ has infinitely long minimizing geodesics in every direction through the base point x_∞. This means that $(N, \bar{g}_\infty)$ contains a line in every direction through x_∞.

Now the well-known Cheeger-Gromoll splitting theorem [15] states that a complete manifold with non-negative Ricci curvature splits isometrically along any line. It follows that $(N, \bar{g}_\infty)$ splits isometrically in every direction through x_∞, and hence $(N, \bar{g}_\infty) = (\mathbb{R}^n, g_0)$, where g_0 is the flat metric on $\mathbb{R}^n$.

Now of course $(\mathbb{R}^n, g_0)$ has infinite harmonic radius. If the convergence of $(N, \bar{g}_i)$ to the limit $(\mathbb{R}^n, g_0)$ can be shown to be in the strong $L^{2,p}$ topology, then the continuity of r_h in this topology immediately gives a contradiction, since by (2.6), the limit $(N, \bar{g}_\infty)$ has $r_h(x_\infty) = 1$.

The second or analytic part of the argument is to prove strong $L^{2,p}$ convergence to the limit. The idea here is to use elliptic regularity to bootstrap or improve the smoothness of the convergence.

In harmonic coordinates, the Ricci curvature of a metric g has the following especially simple form:

$$-\frac{1}{2}\Delta g_{\alpha\beta} + Q_{\alpha\beta}(g, \partial g) = Ric_{\alpha\beta}, \tag{2.9}$$

where $\Delta = g^{\alpha\beta}\partial_\alpha\partial_\beta$ is the Laplacian w.r.t. the metric g and Q is quadratic in g, its inverse, and ∂g. In particular, if $r_h(x) = 1$ and $r_h(y) \geq r_o > 0$, for all $y \in \partial B_x(1)$, then one has a uniform $L^{1,p}$ bound on Q and uniform $L^{2,p}$ bounds on the coefficients for the Laplacian within $B_x(1 + \frac{1}{2}r_o)$.

If now Ric is uniformly bounded in L^∞, then standard elliptic regularity applied to (2.9) implies that $g_{\alpha\beta}$ is uniformly controlled in $L^{2,q}$, for any $q < \infty$, (in particular for $q > p$). More importantly, if g_i is a sequence of metrics for which $(Ric_{g_i})_{\alpha\beta}$ converges strongly in L^p to a limit $(Ric_{g_\infty})_{\alpha\beta}$, then elliptic regularity again implies that the metrics $(g_i)_{\alpha\beta}$ converge strongly in

$L^{2,p}$ to the limit $(g_\infty)_{\alpha\beta}$. For the metrics $\bar{g}_i$, (2.7) implies that $Ric \to 0$ in L^∞, and so $Ric \to 0$ strongly in L^q, for any $q < \infty$.

These remarks essentially prove that the $L^{2,p}$ harmonic radius is continuous w.r.t. the strong $L^{2,p}$ topology. Further, when applied to the sequence $\bar{g}_i$ and using (2.6), they imply that the metrics $\bar{g}_i$ converge strongly in $L^{2,p}$ to the limit $\bar{g}_\infty$. This completes the proof.

It is easy to see from the proof that the lower bound on the injectivity radius in (2.3) can be considerably weakened. For instance, define the 1-cross $Cro_1(x)$ of (M, g) at x to be the length of the longest minimizing geodesic in (M, g) with center point x and set

$$Cro_1(M, g) = \inf_x Cro_1(x).$$

We introduce this notion partly because it has a natural analogue in Lorentzian geometry, when a minimizing geodesic is replaced by a maximizing time-like geodesic, cf. §5. Then one has the following result on 4-manifolds, cf. [4].

Theorem 2.3. (Convergence II). *Let M be a 4-manifold. Then the conclusions of Theorem 2.1 hold under the bounds*

$$|Ric| \leq k, \ Cro_1 \geq c_o, \ vol \geq v_o, \ diam \leq D. \tag{2.10}$$

The proof is the same as that of Theorem 2.2. The lower bound on Cro_1 implies that on the blow-up limit $(N, \bar{g}_\infty, x_\infty)$ above, one has *a* line. Hence, the splitting theorem implies that $N = N' \times \mathbb{R}$. It follows that N' is Ricci-flat and hence, since $dim N' = 3$, N' is flat. Using the volume bound in (2.10), it follows that $(N, \bar{g}_\infty) = (\mathbb{R}^4, g_0)$, cf. (2.12)-(2.13) below. (The volume bound rules out the possibility that N' is a non-trivial flat manifold of the form $\mathbb{R}^3/\Gamma$). This gives the same contradiction as before.

Of course, in dimension 3 any Ricci-flat manifold is necessarily flat, and so the same proof shows that one has $C^{1,\alpha}$ and $L^{2,p}$ precompactness within the class of metrics on 3-manifolds satisfying

$$|Ric| \leq k, \ vol \geq v_o, \ diam \leq D. \tag{2.11}$$

Remark 2.4. (i). Although (2.4) gives the existence of a lower bound on r_h in terms of the bounds k, i_o and D, currently there is no proof of an effective or computable bound. Equivalently, there is no direct proof of Theorem 2.1, which does not involve a passage to limits and invoking a contradiction. This is closely related to the fact there is currently no *quantitative* or *finite* version of the Cheeger-Gromoll splitting theorem, where one can deduce definite bounds on the metric in the presence of (a collection of) minimizing geodesics of a finite but definite length.

If however the bound on $|Ric|$ in (2.3) is strengthened to a bound on $|Riem|$, as in (1.4), then it is not difficult to obtain an effective or computable lower bound on r_h, cf. [32].

(ii). The proof above can be easily adapted to give a similar result if the L^∞ bound on Ric is replaced by an L^q bound, for some $q > n/2$; one then obtains convergence in weak $L^{2,q}$.

In the opposite direction, the convergence can be improved if one has bounds on the derivatives of the Ricci curvature. This will be the case if Ric satisfies an elliptic system of PDE, for instance the Einstein equations. In this case, one obtains C^∞ convergence to the limit.

(iii). The assumption that M is closed in Theorem 2.2 is merely for convenience, and an analogous result holds for open manifolds, away from the boundary.

The bounds on injectivity radius in (2.3), or even the 1-cross in (2.10), are rather strong and one would like to replace them with merely a lower volume bound, as in (2.11).

An elementary but important result, the volume comparison theorem of Bishop-Gromov [27], states that if $Ric \geq (n-1)k$, for some k, on (M, g), $n = dimM$, then the ratio

$$\frac{volB_x(r)}{volB_k(r)} \downarrow \tag{2.12}$$

is monotone non-increasing in r; here $volB_k(r)$ is the volume of the geodesic r-ball in the n-dimensional space form of constant curvature k. In particular, if the bounds (2.11) hold, in dimension n, then (2.12) gives a lower bound on the volumes of balls on *all* scales:

$$volB_x(r) \geq \frac{volM}{volB_k(D)} \cdot volB_k(r). \tag{2.13}$$

Note that the estimate (2.13) also implies that, for any fixed $r > 0$, if $volB_x(r) \geq v_0 > 0$, then $volB_y(r) \geq v_1 > 0$, where v_1 depends only on v_0 and $dist_g(x, y)$. Thus, the ratio of the volumes of unit balls cannot become arbitrarily large or small on domains of bounded diameter.

Now a classical result of Cheeger [14] implies that if (2.11) is strengthened to

$$K_P \geq -K, \ vol \geq v_o, \ diam \leq D, \tag{2.14}$$

where K_P is the sectional curvature of any plane P in the tangent bundle TM, then one has a lower bound on the injectivity radius, $inj_g(M) \geq i_o(K, v_o, D)$. However, this estimate fails under the bounds (2.11), cf. [2]. It is worthwhile to exhibit a simple concrete example illustrating this.

Example 2.5. Let g_λ be the family of Eguchi-Hanson metrics on the tangent bundle TS^2 of S^2. The metrics g_λ are given explicitly by

$$g_\lambda = [1 - (\frac{\lambda}{r})^4]^{-1} dr^2 + r^2[1 - (\frac{\lambda}{r})^4]\theta_1^2 + r^2(\theta_2^2 + \theta_3^2). \tag{2.15}$$

Here $\theta_1, \theta_2, \theta_3$ are the standard left-invariant coframing of $SO(3) = \mathbb{RP}^3$, (the sphere bundles in TS^2) and $r \geq \lambda$. The locus $r = \lambda$ is the image of the 0-section and is a totally geodesic round $S^2(\lambda)$ of radius λ.

The metrics g_λ are Ricci-flat, and are all homothetic, i.e. are rescalings (via diffeomorphisms) of a fixed metric; in fact,

$$g_\lambda = \lambda^2 \cdot \psi_\lambda^*(g_1), \tag{2.16}$$

where $\psi_\lambda(r) = \lambda r$, and ψ_λ acts trivially on the $SO(3)$ factor. As $\lambda \to 0$, i.e. as one blows down the metrics, g_λ converges to the metric g_0, the flat metric on the cone $C(\mathbb{RP}^3)$. The convergence is smooth in the region $r \geq r_o$, for any fixed $r_o > 0$, but is not smooth at $r = 0$. Since $S^2(\lambda)$ is totally geodesic, the injectivity radius at any point of $S^2(\lambda)$ is $2\pi\lambda$, which tends to 0. On the other hand, the volumes of unit balls, or balls of any definite radius, remain uniformly bounded below.

One sees here that the metrics (TS^2, g_λ) converge as $\lambda \to 0$ to a limit metric on a singular space $C(\mathbb{RP}^3)$. The limit is an orbifold $\mathbb{R}^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection in the origin.

The Eguchi-Hanson metric is the first and simplest example of a large class of Ricci-flat ALE (asymptotically locally Euclidean) spaces, whose metrics are asymptotic to cones $C(S^3/\Gamma)$, $\Gamma \subset SO(4)$, on spherical space forms. This is the family of ALE gravitational instantons, studied in detail by Gibbons and Hawking, cf. [26] and references therein, in connection with Hawking's Euclidean quantum gravity program.

It is straightforward to modify the construction in Example 2.5 to obtain orbifold degenerations on compact 4-manifolds satisfying the bounds (2.11). Thus, one does not have $C^{1,\alpha}$ or even C^0 (pre)-compactness of the space of metrics on M under the bounds (2.11). Singularities can form in passing to limits, although the singularities are of a relatively simple kind. The next result from [1] shows that this is the only kind of possible degeneration or singularity formation.

Theorem 2.6. (Convergence III). *Let $\{g_i\}$ be a sequence of metrics on a 4-manifold, satisfying the bounds*

$$|Ric| \leq k, \;\; vol \geq v_o, \;\; diam \leq D. \tag{2.17}$$

Then a subsequence converges, (in the Gromov-Hausdorff topology), to an orbifold (V, g), with a finite number of singular points $\{q_j\}$. Each singular point q has a neighborhood homeomorphic to a cone $C(S^3/\Gamma)$, for Γ a finite subgroup of $SO(4)$.

The metric g is $C^{1,\alpha}$ or $L^{2,p}$ on the regular set

$$V_0 = V \setminus \cup\{q_j\},$$

and extends in a local uniformization of a singular point to a C^0 Riemannian metric. Further, there are embeddings

$$F_i : V_0 \to M$$

such that $F_i^*(g_i)$ converges in the $C^{1,\alpha}$ topology to the metric g.

Here, convergence in the Gromov-Hausforff topology means convergence as metric spaces, cf. [27]. We mention only a few important issues in the proof of Theorem 2.6. First, the Chern-Gauss-Bonnet formula implies that for metrics with bounded Ricci curvature and volume on 4-manifolds, one has an apriori bound on the L^2 norm of the full curvature tensor:

$$\tfrac{1}{8\pi^2}\int_M |R|^2 dV \leq \chi(M) + C(k, V_o),$$

where $C(k, V_o)$ is a constant depending only on k from (2.17) and an upper bound V_o on $vol_g M$: $\chi(M)$ is the Euler characteristic of M. Second, with each singular point $q \in V$, there is a associated a sequence of rescalings $\bar{g}_i = \lambda_i^2 g_i, \lambda_i \to \infty$, and base points $x_i \in M$, $x_i \to q$, such that a subsequence of $(M, \bar{g}_i, x_i)$ converges in $C^{1,\alpha} \cap L^{2,p}$ to a non-trivial Ricci-flat ALE space $(N, \bar{g}_\infty)$ as above. It is not difficult to see that any such ALE space has a definite amount of curvature in L^2. This implies basically that there are only a finite number of such singular points. Further, the ALE spaces N are embedded in M, in a topologically essential way.

3. Collapse/Formation of Cusps.

In this section, we consider what happens when

$$vol \to 0 \quad \text{or} \quad diam \to \infty$$

in the bounds (2.11). This involves the notion of Cheeger-Gromov collapse, or collapse with bounded curvature.

For simplicity, we restrict the discussion to dimension 3. While there is a corresponding theory in higher dimensions, cf. [16], there are special and advantageous features that hold only in dimension 3. Further, the relations with general relativity are most direct in dimension 3, in that the discussion can be applied to the behavior of space-like hypersurfaces in a given space-time.

The simplest non-trivial example of collapse is the Berger collapse of the 3-sphere along S^1 fibers of the Hopf fibration. Thus, consider the family of metrics on S^3 given by

$$g_\lambda = \lambda^2\theta_1^2 + (\theta_2^2 + \theta_3^2), \tag{3.1}$$

where $\theta_1, \theta_2, \theta_3$ are the standard left-invariant coframing of S^3. The metrics g_λ have an isometric S^1 action, with Killing field K dual to θ_1, with length of the S^1 orbits given by $2\pi\lambda$. Thus, in letting $\lambda \to 0$, one is blowing down the metric in *one* direction. (This is exactly what occurs on approach to the horizon of the Taub-NUT metric, cf. [31]). A simple calculation shows that the curvature of g_λ remains uniformly bounded as $\lambda \to 0$. Clearly $vol_{g_\lambda} S^3 \sim \lambda \to 0$. The metrics g_λ collapse S^3 to a limit space, in this case S^2.

This same procedure may be carried out, with the same results, on any 3-manifold (or n-manifold) which has a free or locally free isometric S^1

action; locally free means that the isotropy group of any orbit is a finite subgroup of S^1, i.e. there are no fixed points of the action. Similarly, one may collapse along the orbits, as in (3.1), of a locally free T^k-action, where T^k is the k-torus. Remarkably, Gromov [28] showed that more generally one may collapse along the orbits of an isometric nilpotent group action, and furthermore, such groups are *only* groups which allow such a collapse with bounded curvature. Thus for instance collapsing along the orbits of an isometric G-action, where G is semi-simple and non-abelian, increases the curvature without bound.

A 3-manifold which admits a locally free S^1 action is called a Seifert fibered space. Such a space admits a fibration over a surface V, with S^1 fibers. Where the action is free, this fibration is a circle bundle. There may exist an isolated collection of non-free orbits, corresponding to isolated points in V. Topologically, a neighborhood of such an orbit is of the form $D^2 \times S^1$, where the S^1 acts by rotation on the S^1 factor and by rotation through a rational angle about $\{0\}$ in D^2.

The collection of Seifert fibered spaces falls naturally into 6 classes, according to the topology of the base surface V, i.e. $V = S^2$, T^2, or Σ_g, $g \geq 2$, and according to whether the S^1 bundle is trivial or not trivial. These account for 6 of the 8 possible geometries of 3-manifolds in the sense of Thurston [39]; these geometries are: $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$, S^3, Nil, and $\widetilde{SL(2, \mathbb{R})}$, respectively. The two remaining geometries are Sol, corresponding to non-trivial torus bundles over S^1, and the hyperbolic geometry $\mathbb{H}^3$.

Now suppose N is a compact Seifert fibered space with boundary. The boundary is a finite collection of tori, on which one has a free S^1 action. In a neighborhood of the boundary, this S^1 action then embeds in the standard free T^2 action on $T^2 \times I$. Given a collection of such spaces N_i, one may then glue the toral boundaries together by automorphisms of the torus, i.e. by elements in $SL(2, \mathbb{Z})$. For example, the glueing may interchange the fiber and base circles.

Definition 3.1. *A* graph manifold *G is a 3-manifold obtained by glueing Seifert fibered spaces by toral automorphisms of the boundary tori.*

Thus, a graph manifold has a decomposition into two types of regions,

$$G = S \cup L. \tag{3.2}$$

Each component of S is a Seifert fibered space, while each component of L is $T^2 \times I$, and glues together different boundary components of elements in S. The exceptional case of glueing two copies of $T^2 \times I$ by toral automorphisms of the boundary is also allowed; this defines the class of Sol manifolds, up to finite covers. The Seifert fibered components have a locally free S^1 action, the $T^2 \times I$ components have a free T^2 action; in general, these group actions do not extend to actions on topologically larger domains.

Graph manifolds are an especially simple class of 3-manifolds; one has a complete understanding of their topological classification [41]. The terminology comes from the fact that one may associate a graph to G, by assigning a vertex to each component of S, and an edge to each component of L which connects a pair of components in S.

It is not difficult to generalize the construction above to show that any closed graph manifold G admits a sequence of metrics g_i which collapse with uniformly bounded curvature, i.e.

$$|Ric_{g_i}| \leq k, \quad vol_{g_i} G \to 0. \tag{3.3}$$

The metrics g_i collapse the Seifert fibered pieces along the S^1 orbits, while collapsing the toral regions $T^2 \times I$ along the tori. Thus the collapse is rank 1 along S, while rank 2 along L. (Of course a bound on the full curvature is the same as a bound on the Ricci curvature in dimension 3).

If the graph manifold is Seifert fibered, then the collapse (3.3) may be carried out with bounded diameter,

$$diam_{g_i} S \leq D, \text{ for some } D < \infty. \tag{3.4}$$

In fact, if S is a Nil-manifold, then the collapse may be carried out so that $diam_{g_i} S \to 0$, cf. [28].

On the other hand, suppose G is non-trivial in that it has both S and L components. If N denotes any S or L component, then necessarily

$$diam_{g_i} N \to \infty \tag{3.5}$$

under the bounds (3.3). (This phenomenon can be viewed as a refinement of the remark following (2.13)). In particular, the transition from Seifert fibered domains to toral domains takes longer and longer distance the more collapsed the metrics are. One obtains different collapsed "limits" depending on choice of base point. This "pure" behavior on regions of bounded diameter is special to dimension 3.

The Cheeger-Gromov theory, [16], implies that the converse also holds.

Theorem 3.2. (Collapse). *If M is a closed 3-manifold which collapses with bounded curvature, i.e. there is a sequence of metrics such that (3.3) holds, then M is a graph manifold.*

In fact, this result holds if M admits a sufficiently collapsed metric, i.e. $|Ric_g| \leq k$ and $vol_g M \leq \varepsilon_o$, for some $\varepsilon = \varepsilon_o(k)$ sufficiently small. Note of course that a collapsing sequence of metrics g_i is not *necessarily* invariant under the (merely smooth) S^1 or T^2 actions associated with the graph manifold structure.

In a certain sense, the vast majority of 3-manifolds are not graph manifolds, and so Theorem 3.2 gives strong topological restrictions on the existence of sufficiently collapsed metrics.

Idea of proof: First, it is easy to see that

$$vol_{g_i}B_x(1) \to 0 \Rightarrow inj_{g_i}(x) \to 0.$$

At any x, rescale the metrics g_i to make $inj(x) = 1$, i.e. set

$$\bar{g}_i = [inj_{g_i}(x)]^{-2} \cdot g_i.$$

Now the bound (3.3) gives $|Ric_{\bar{g}_i}| \sim 0$. Thus, the metrics $\bar{g}_i$ are close to flat metrics on $\mathbb{R}^3/\Gamma$, where Γ is a non-trivial discrete group of Euclidean isometries, (by Theorem 2.2 for instance). Thus, essentially, $\mathbb{R}^3/\Gamma = \mathbb{R}^2 \times S^1$, or $\mathbb{R} \times S^1 \times S^1$. It follows that the local geometry, i.e. the geometry on the scale of the injectivity radius, is modeled by *non-trivial, flat* 3-manifolds. One then shows that these local structures for the geometry and topology can be glued together consistently to give a global graph manifold structure.

If S is a Seifert fibered space, possibly with boundary ∂S, the orbits of the S^1 action always inject in $\pi_1(S)$, i.e.

$$\pi_1(S^1) \hookrightarrow \pi_1(S),$$

unless $S = S^3/\Gamma$, or in the case of boundary, $S = D^2 \times S^1$, cf. [36]. Thus, if a graph manifold G is not a spherical space form, or does not have a solid torus component in its Seifert fibered decomposition (3.2), then the fibers of the decomposition, namely circles and tori, always inject in π_1:

$$\pi_1(fiber) \hookrightarrow \pi_1(G). \tag{3.6}$$

Hence, in this situation, one can pass to covering spaces to *unwrap* any collapse. Thus, if g_i is a collapsing sequence of metrics, by passing to larger and larger covering spaces, based sequences will always have convergent subsequences (in domains of arbitrary but bounded diameter). In addition, the isometric covering transformations on the covers have displacement functions converging uniformly to 0 on compact subsets. Hence, all such limits have a free isometric S^1 or T^2 action, depending on whether the collapse is rank 1 or 2 on the domains. This means that the limits have an extra symmetry not necessarily present on the initial collapsing sequence. Again, this feature of being able to unwrap collapse by passing to covering spaces is special to dimension 3.

Finally, we discuss the third possibility, the formation of cusps. This case, although the most general, corresponds to a mixture of the two previous cases convergence/collapse, and so no essentially new phenomenon occurs. To start, given a complete Riemannian manifold (M, g), choose $\varepsilon > 0$ small, and let

$$M^{\varepsilon} = \{x \in M : volB_x(1) \geq \varepsilon\}, \ M_{\varepsilon} = \{x \in M : volB_x(1) \leq \varepsilon\}. \tag{3.7}$$

M^{ε} is called the ε-thick part of (M, g), while M_{ε} is the ε-thin part.

Now suppose g_i is a sequence of complete Riemannian metrics on the manifold M.

• If $x_i \in M^{\varepsilon}$, for some fixed $\varepsilon > 0$, then one has convergence, (in subsequences), in domains of arbitrary but bounded diameter about $\{x_i\}$, cf. (2.13ff). Essentially, the bounds (2.11) hold on such domains in this case.
• If $y_i \in M_{\varepsilon_o}$, for ε_o sufficiently small, then domains of bounded, depending on ε_o, diameter about $\{y_i\}$ are graph manifolds, in fact Seifert fibered spaces.
• If $z_i \in M_{\varepsilon_i}$, $\varepsilon_i \to 0$, then domains of arbitrary but bounded diameter about $\{z_i\}$ are collapsing.

If $(M_{\varepsilon}, g_i) = \emptyset$, for some fixed $\varepsilon > 0$, then one is in the convergence situation. If $(M^{\varepsilon}, g_i) = \emptyset$, for all $\varepsilon > 0$ sufficiently small, depending on i, then one is in the collapsing situation. The only remaining possibility is that there exist points x_i and y_i in M such that, for any fixed $\varepsilon > 0$,

$$(M^{\varepsilon}, g_i) \neq \emptyset, \text{ and } (M_{\varepsilon}, g_i) \neq \emptyset. \tag{3.8}$$

This is equivalent to the existence of base points x_i, y_i, such that,

$$volB_{x_i}(1) \geq \varepsilon_1, \quad volB_{y_i}(1) \to 0, \tag{3.9}$$

for some $\varepsilon_1 > 0$. Observe that the volume comparison theorem (2.13) implies that $dist_{g_i}(x_i, y_i) \to \infty$ as $i \to \infty$, so that these different behaviors become further and further distant as $i \to \infty$.

This leads to the following result, cf. [5], [16] for further details.

Theorem 3.3. (Cusp Formation). *Let M be a 3-manifold and g_i a sequence of unit volume metrics on M satisfying (3.8). Then pointed subsequences $(M,\, g_i, p_i)$ converge to one of the following:*
• *complete cusps $(N, g_{\infty}, p_{\infty})$. These are complete, open Riemannian 3-manifolds, of finite volume and with graph manifold ends, which collapse at infinity. The convergence in $C^{1,\alpha}$ and weak $L^{2,p}$ topologies, uniform on compact subsets.*
• *Collapsed graph manifolds of infinite diameter.*

In contrast to the topological implications of collapse in Theorem 3.2, (i.e collapse implies M is a graph manifold), in general there are no apriori topological restrictions on M imposed by Theorem 3.3. To illustrate, let M be an arbitrary closed 3-manifold and let $\{C_k\}$ be a collection of disjoint solid tori $D^2 \times S^1$ embedded in M; for example $\{C_k\}$ may be a tubular neighborhood of a (possibly trivial) link in M. Then it is not difficult to construct a sequence of metrics of bounded curvature which converge to a collection of complete cusps on $M \setminus \cup C_k$ and collapse along the standard graph manifold structure on each C_k.

The ends of the cusp manifolds N in Theorem 3.3, i.e. the graph manifolds, necessarily have embedded tori. If such tori are essential in M, i.e. inject on the π_1 level, then Theorem 3.3. does imply strong topological constraints on the topology of M; cf. §6 for some further discussion.

Remark 3.4. We point out that there are versions of Theorems 3.2 and 3.3 also in dimension 4, as well as in higher dimensions. The concept of graph manifold is generalized to manifolds having an "F-structure", or an

"N-structure" (F is for flat, N is for nilpotent), cf. [16], provided bounds are assumed on the full curvature, as in (1.4). In dimension 4, this can be relaxed to bounds on the Ricci curvature, as in (1.5), provided one allows for a finite number of singularities in F-structure, as in Theorem 2.6.

4. Applications to Static and Stationary Space-Times.

In this section, we discuss applications of the results of §2-3 to static and stationary space-times, i.e. space-times (**M, g**) which admit a time-like Killing field K. These space-times are viewed as being the end or final state of evolution of a (time dependent) gravitational field. Since they are time-independent in a natural sense, they may be analysed by methods of Riemannian geometry, which are not available in general for Lorentzian manifolds.

Throughout this section, we assume that (**M, g**) is chronological, i.e. (**M, g**) has no closed time-like curves, and that K is a complete vector field.

Let Σ be the orbit space of the isometric $\mathbb{R}$-action generated by the Killing field K, and let $\pi : \mathbf{M} \to \Sigma$ be the projection to the orbit space. The 4-metric $\mathbf{g}$ has the form

$$\mathbf{g} = -u^2(dt + \theta)^2 + \pi^*(g), \tag{4.1}$$

where $K = \partial/\partial t, \theta$ is a connection 1-form for the bundle π, $u^2 = -\mathbf{g}(K, K) > 0$ and $g = g_\Sigma$ is the metric induced on the orbit space.

The vacuum Einstein equations are equivalent to an elliptic system of P.D.E's in the data (Σ, g, u, θ). Let ω be the twist 1-form on Σ, given by $2\omega = *(\kappa \wedge d\kappa) = -u^4 d\theta$, where $\kappa = -u^2(dt + \theta)$ is the 1-form dual to K. Then the equations on Σ are:

$$Ric_g = u^{-1}D^2u + 2u^{-4}(\omega \otimes \omega - |\omega|^2 g), \tag{4.2}$$

$$\Delta u = -2u^{-3}|\omega|^2, \tag{4.3}$$

$$d\omega = 0. \tag{4.4}$$

The maximum principle applied to (4.3) immediately implies that if Σ is a closed 3-manifold, then (Σ, g) is flat and $u = const$, and so (**M, g**) is a (space-like) isometric quotient of empty Minkowski space $(\mathbb{R}^4, \eta)$. Thus, we assume Σ is open, possibly with boundary.

Locally of course there are many solutions to the system (4.2)-(4.4); to obtain uniqueness, one needs to impose boundary conditions.

We consider first the global situation, and so assume that (Σ, g) is a complete, non-compact Riemannian 3-manifold. Boundary conditions are then at infinity, i.e. conditions on the asymptotic behavior of the metric. In this respect, one has the following classical result, c.f. [33], [22].

Theorem 4.1. (Lichnerowicz). *The only complete, stationary vacuum space-time (*$\mathbf{M},\ \mathbf{g}$*) which is asymptotically flat (AF) is empty Minkowski space-time* $(\mathbb{R}^4, \eta)$.

It is most always taken for granted that Σ should be AF. Stationary space-times are meant to model isolated physical systems, and the only physically realistic models are AF. In fact, from this physical perspective, the Lichnerowicz theorem may be viewed as a triviality. Since there is no source for the gravitational field, it must be the empty Minkowski space-time.

However, mathematically, the Lichnerowicz theorem is not (so) trivial, and if it were false one would be forced to revise physical intuition. Moreover, the assumption that $(\mathbf{M},\ \mathbf{g})$ is AF is contrary to the spirit of general relativity. Such a boundary condition is adhoc, and its imposition is in fact circular in a certain sense. Apriori, there might well be complete stationary solutions for which the curvature does not decay anywhere to 0 at infinity. The following result from [6] clarifies this issue.

Theorem 4.2. (Generalized Lichnerowicz). *The only complete stationary vacuum space-time (*$\mathbf{M},\ \mathbf{g}$*) is empty Minkowski space-time* $(\mathbb{R}^4, \eta)$*, or a discrete isometric quotient of it.*

The starting point of the proof of this result is to study first the moduli space of all complete stationary vacuum solutions. As noted above, apriori any given solution may, apriori, have unbounded curvature, i.e. $|Ric_g|$ may diverge to infinity on divergent sequences in Σ. Under such a condition, the first step is then to show, by taking suitable base points and rescalings, that one may then obtain a new stationary vacuum solution, (i.e. a new point in the moduli space), with uniformly bounded curvature, and non-zero curvature at a base point. This step uses the Cheeger-Gromov theory, as described in §2-§3, and requires the special features of collapse in 3-dimensions.

The next step in the proof is to recast the problem in the Ernst formulation. Define the conformally related metric $\widetilde{g}$ by

$$\widetilde{g} = u^2 g. \tag{4.5}$$

A simple calculation shows that (4.2) becomes

$$Ric_{\widetilde{g}} = 2(d\ln u)^2 + 2u^{-4}\omega^2 \geq 0. \tag{4.6}$$

Further, the system (4.2)-(4.4) becomes the Euler-Lagrange equations for an effective 3-dimensional action given by

$$\mathcal{S}_{eff} = \int [R - \frac{1}{2}(\frac{|d\phi|^2 + |du^2|^2}{u^4})]dV.$$

Here ϕ is the twist potential, given by $d\phi = 2\omega$. (In general one must pass to the universal cover to obtain the existence of ϕ).

This action is exactly 3-dimensional (Riemannian) gravity on $(\Sigma, \widetilde{g})$ coupled to a σ-model with target the hyperbolic plane $(H^2(-1), g_{-1})$. Thus, the Ernst map $E = (\phi, u^2)$ is a harmonic map

$$E : (\Sigma, \widetilde{g}) \to (H^2(-1), g_{-1}). \tag{4.7}$$

Now it is well-known that harmonic maps $E : (M, g) \to (N, h)$ from Riemannian manifolds of non-negative Ricci curvature to manifolds of non-positive sectional curvature have strong rigidity properties, via the Bochner-Lichnerowicz formula,

$$\frac{1}{2}\Delta|DE|^2 = |D^2E|^2 + < Ric_g, E^*(h) > - \sum(E^*R_h)(e_i, e_j, e_j, e_i). \tag{4.8}$$

By analysing (4.8) carefully, one shows that E is a constant map, from which it follows easily that $(\mathbf{M}, \mathbf{g})$ is flat.

Remark 4.3. (i). The same result and proof holds for stationary gravitational fields coupled to σ-models, whose target spaces are Riemannian manifolds of non-positive sectional curvature, i.e. $E : (\Sigma, \widetilde{g}) \to (N, g_N)$ with $Riem_{g_N} \leq 0$.

(ii). Curiously, the Riemannian analogue of Theorem 4.2 remains an open problem. Thus, does there exist a complete non-flat Ricci-flat Riemannian 4-manifold which admits a free isometric S^1 action?

(iii). It is interesting to note that the analogue of Theorem 4.2 is false for stationary Einstein-Maxwell solutions. A counterexample is provided by the (static) Melvin magnetic universe [34], cf. also [25]. I am grateful to David Garfinkle for pointing this out to me. For the stationary Einstein-Maxwell system, the target space of the Ernst map is $SU(2, 1)/S(U(1, 1) \times U(1))$, $(SO(2, 1)/SO(1, 1)$ for static Einstein-Maxwell). Both of these target spaces have indefinite, (i.e. non-Riemannian), metrics.

The rigidity result Theorem 4.2 leads to apriori estimates on the geometry of general stationary solutions of the Einstein equations, cf. [6]. Thus, if Σ is not complete, it follows that $\partial\Sigma \neq \emptyset$. Note that part of $\partial\Sigma$ may correspond to the horizon $H = \{u = 0\}$ where the Killing field vanishes.

Theorem 4.4. (Curvature Estimate). *Let $(\mathbf{M}, \mathbf{g})$ be a stationary vacuum space-time. Then there is a constant $C < \infty$, independent of $(\mathbf{M}, \mathbf{g})$, such that*

$$|\mathbf{R}|(x) \leq C/r^2[x], \tag{4.9}$$

where $r[x] = dist_\Sigma(\pi(x), \partial\Sigma)$.

Here, the curvature norm $|\mathbf{R}|$ may be given by

$$|\mathbf{R}|^2 = |R_\Sigma|^2 + |d \ln u|^2 + |u^{-2}\omega|^2.$$

Note that Theorem 4.2 follows from Theorem 4.4 by letting $r \to \infty$. Conversely, it is a general principle for elliptic geometric variational problems that a global rigidity result as in Theorem 4.2 leads to apriori local estimates as in Theorem 4.4.

Remark 4.5. (i). Using elliptic regularity, one also has higher order bounds:

$$|\nabla^k \mathbf{R}|(x) \leq C_k/r^{2+k}[x]. \tag{4.10}$$

(ii). A version of this result also holds for stationary space-times with energy-momentum tensor T. Thus, for example one has

$$|\mathbf{R}|(x) \leq C_\alpha \cdot |T|_{C^\alpha(B_{[x]}(1))}, \tag{4.11}$$

for any $\alpha > 0$, where $B_{[x]}(1)$ is the unit ball in (Σ, g) about $[x]$. The proof is the same as that of (4.9) given in [6].

Thus, one can use the Cheeger-Gromov theory to control the local behavior of stationary space-times, possibly with matter terms, away from any boundary.

The results above can in turn be applied to study the possible asymptotic behavior of general stationary or static vacuum space-times, without any apriori AF assumption. For example, (4.9) implies that the curvature decays at least quadratically in any end (E, g) of (Σ, g). For simplicity, we restrict to static space-times.

Thus, let $(\mathbf{M}, \mathbf{g})$ be a static space-time with orbit space (Σ, g), with $\partial\Sigma \neq \emptyset$. Define $\partial\Sigma$ to be *pseudo-compact* if there exists $r_o > 0$ such that the level set $\{r = r_o\}$ in Σ is compact; recall that r is the distance function to the boundary $\partial\Sigma$. (There are numerous examples of static space-times for which $\partial\Sigma$ is non-compact, with $\partial\Sigma$ pseudo-compact). Let $S(r) = r^{-1}(r) \subset \Sigma$. If E is an end of (Σ, g), define its mass m_E by

$$m_E = lim_{s\to\infty} \frac{1}{4\pi} \int_{S(s)} < \nabla \ln u, \nabla t > dA. \tag{4.12}$$

It is easily seen from the static vacuum equations that the integral is monotone non-increasing in s, and so the limit exists. The mass m_E coincides with the Komar mass in case E is AF. The following result is from [7].

Theorem 4.6. (Static Asymptotics). *Let $(\mathbf{M}, \mathbf{g}, u)$ be a static vacuum space-time with pseudo-compact boundary. Then $(\mathbf{M}, \mathbf{g})$ has a finite number of ends. Any end E on which*

$$liminf_E u > 0, \tag{4.13}$$

is either:

$$AF$$

or

$$\text{small} \equiv_{def} \int_1^\infty areaS(r)^{-1} dr < \infty. \tag{4.14}$$

Further, if $m_E \neq 0$ and $sup_E u < \infty$, then E is AF.

This result is sharp in the sense that if any of the hypotheses are dropped, then the conclusion is false. For instance, if (4.13) fails, then there are examples of static vacuum solutions with ends neither small nor AF.

We note that when E is AF, the result implies it is AF in the strong sense that

$$(4.15)\quad |g - g_0| = \frac{2m}{r} + O(r^{-2}),\ |R| = O(r^{-3}),\ \text{and}\ |u-1| = \frac{m}{r} + O(r^{-2}).$$

More precise asymptotics can then be obtained by using standard elliptic estimates on the equations (4.2)-(4.4), or from [12]. Again, a version of Theorem 4.6 holds for static space-times with matter, cf. again [7] for further information.

The idea of the proof is to study the asymptotic behavior of an end E by blowing it down, as described in §1. Thus, for R large and any fixed k, consider the metric annuli $A(R, kR)$ about some base point $x_o \in (\Sigma, g)$ and consider the rescalings $g_R = R^{-2}g$. The annulus $A(R, kR)$ then becomes an annulus of the metric form $A(1, k)$ w.r.t. g_R. Further, the estimate (4.9) implies that the curvature of g_R in $A(1, k)$ is uniformly bounded. Thus, one may apply the Cheeger-Gromov theory as described in §2,§3, to a sequence $(A(1, k), g_{R_i})$, with $R_i \to \infty$. One proves that the convergence case gives rise to AF ends, while the collapse case gives rise to small ends.

Note that in the collapsing situation, one obtains an extra S^1 or T^2 symmetry when the collapse is unwrapped in covering spaces. Thus, the behavior in this case is described by axisymmetric static solutions, i.e. the Weyl metrics. Small ends typically have the same end structure as $\mathbb{R}^2 \times S^1$, where the S^1 factor has bounded length and so typically have at most quadratic growth for the area of geodesic spheres.

It is worth pointing out that there are static vacuum solutions, smooth up to a compact horizon, which have a single small end. This is the family of Myers metrics [35], or periodic Schwarzschild metrics, (discovered later and independently by Korotkin and Nicolai). The manifold Σ is topologically $(D^2 \times S^1) \setminus B^3$, so that $\partial\Sigma = S^2$ with a single end of the form $T^2 \times \mathbb{R}^+$. Metrically, the end is asymptotic to one of the (static) Kasner metrics. This is of course not a counterexample to the static black hole uniqueness theorem, since the end is not AF.

Note that since $\pi_1(\Sigma) = \mathbb{Z}$ here, one may take non-trivial covering spaces of the Myers metrics. This leads to static vacuum solutions with an arbitrary finite number, or even an infinite number, of black holes in static equilibrium. This situation is of course not possible in Newtonian gravity, and so is a highly non-linear effect of general relativity.

5. Lorentzian Analogues and Open Problems.

In this section, we discuss potential analogues of the results of §2 and §3 for Lorentzian metrics on 4-manifolds. The main interest is in space-times $(\mathbf{M}, \mathbf{g})$ for which one has control on the Ricci curvature of $\mathbf{g}$, or via the Einstein equations, control on the energy-momentum tensor T. In particular, the main focus will be on vacuum space-times, $Ric_{\mathbf{g}} = 0$.

One would like to find conditions under which one can take limits of vacuum space-times. One natural reason for trying to do this is the following. There are now a number of situations where global stability results have been proved, namely: the global stability of Minkowski space-time [19], and of deSitter space-time [24], the global future stability of the Milne space-time [10], and the future $U(1)$ stability of $U(1)$ Bianchi models [18]. These results are *openness* results, which state that the basic features of a given model, e.g. Minkowski, are preserved under suitably small perturbations of the initial data. It is then natural to consider what occurs when one tries to pass to limits of such perturbations.

The issue of being able to take limits is also closely related with the existence problem and singularity formation for the vacuum Einstein evolution equations. From this perspective, suppose one has an increasing sequence of domains $(\Omega_i, \mathbf{g_i}), \Omega_i \subset \Omega_{i+1}$ with $\mathbf{g_{i+1}}|_{\Omega_i} = \mathbf{g_i}$, which are evolutions of smooth Cauchy data on some fixed initial data set. If $\mathbf{M} = \cup\Omega_i$ is the maximal Cauchy development, then understanding **(M, g)** amounts to understanding the limiting behavior of $(\Omega_i, \mathbf{g_i})$.

There are two obvious but essential reasons why it is much more difficult to develop a Lorentzian analogue of the Cheeger-Gromov theory, in particular with bounds only on the Ricci curvature. The first is that the elliptic nature of the P.D.E. for Ricci curvature becomes hyperbolic for Lorentz metrics, and hyperbolic P.D.E. are much more difficult than elliptic P.D.E. The second is that the compact group of Euclidean rotations $O(4)$ is compact, while the group of proper Lorentz transformations $O(3,1)$ is non-compact.

A: 1^{st} Level Problem.

Consider first the problem of controlling the space-time metric $\mathbf{g}$ in terms of bounds, say L^∞, on the space-time curvature $\mathbf{R}$,

$$|\mathbf{R}|_{L^\infty} \leq K < \infty, \tag{5.1}$$

since already here there are significant issues.

First, the norm of curvature tensor $|\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl}$ is no longer non-negative for Lorentz metrics, and so a bound on $|\mathbf{R}|^2$ does not imply a bound on all the components $\mathbf{R}_{ijkl}$. In fact, for a Ricci-flat 4-metric, there are exactly two scalar invariants of the curvature tensor:

$$< \mathbf{R}, \mathbf{R} >= |\mathbf{R}|^2 = \mathbf{R}_{ijkl}\mathbf{R}^{ijkl} \text{ and } < \mathbf{R}, *\mathbf{R} >= \mathbf{R}_{ijkl}(*\mathbf{R}^{ijkl}). \tag{5.2}$$

Both of these invariants can vanish identically on classes of Ricci-flat non-flat space-times; for instance this is the case for the class of plane-fronted gravitational waves, given by

$$\mathbf{g} = -dudv + (dx^2 + dy^2) - 2h(u, x, y)du^2,$$

$$\Delta_{(x,y)}h = 0,$$

cf. [13,§8] and references therein. Here, h is only required to harmonic in the variables (x,y), and is arbitrary in u. The class of such space-times is highly non-compact, and so one has no local control of the metric in any coordinate system under bounds on the quantities in (5.2).

Thus, one must turn to bounds on the components of $\mathbf{R}$ in some fixed coordinate system or framing. The most efficient way to do this is to choose a unit time-like vector $T = e_0$, say future directed, and extend it to an orthornormal frame e_α, $0 \leq \alpha \leq 3$. Since the space $T^\perp$ orthogonal to T is space-like and $O(3)$ is compact, the particular framing of $T^\perp$ is unimportant. One may then define the norm w.r.t. T by

$$|\mathbf{R}|_T^2 = \sum(\mathbf{R}_{ijkl})^2, \tag{5.3}$$

where the components are w.r.t. the frame e_α. This is equivalent to taking the norm of $\mathbf{R}$ w.r.t. the Riemannian metric

$$g_E = \mathbf{g} + 2T \otimes T.$$

If, at a given point p, T lies within a compact subset W of the future interior null cone T_p^+, then the norms (5.3) are all equivalent, with constant depending only on W. Of course if D is a compact set in the space-time $(\mathbf{M}, \mathbf{g})$ and the vector field T is continuous in D, then T lies within a compact subset of T^+D, where T^+D is the bundle of future interior null cones in the tangent bundle TD.

It is quite straightforward to prove that if (M,g) is a smooth Riemannian manifold with an L^∞ bound on the full curvature, $|R| \leq K$ then there are local coordinate systems in which the metric is $C^{1,\alpha}$ or $L^{2,p}$, with bounds depending only on K and a lower volume bound, cf. Remark 2.4(i).

However, this has been an open problem for Lorentzian metrics, apparently for some time, cf. [20],[40] for instance. The following result gives a solution to this problem.

To state the result, let Ω be a domain in a smoooth Lorentz manifold $(\mathbf{M}, \mathbf{g})$, of arbitrary dimension $n+1$. Then Ω satisfies the size conditions if the following holds. There is a smooth time function t, with $T = \nabla t/|\nabla t|$ the associated unit time-like vector field on Ω, such that, for $S = S_0 = t^{-1}(0)$, the 1-cylinder

$$C_1 = B_p(1) \times [-1,1] \subset\subset \Omega, \tag{5.4}$$

i.e. C_1 has compact closure in Ω. Here $B_p(1)$ is the geodesic ball of radius 1 about p, w.r.t. the metric g induced on S and the product is identified with a subset of Ω by the flow of T.

It is essentially obvious that any point q in a Lorentz manifold has a neighborhood satisfying the size conditions, when the metric $\mathbf{g}$ is scaled up suitably.

Let $D = ImT|_{C_1} \subset\subset T^+\Omega$.

Theorem 5.1. *Let Ω be a domain in a vacuum $(n+1)$-dimensional space-time $(\mathbf{M}, \mathbf{g})$. Suppose Ω satisfies the size conditions, and that there exist constants $K < \infty$ and $v_o > 0$ such that*

$$|\mathbf{R}|_T \leq K, \quad vol_g B_p(\tfrac{1}{2}) \geq v_o. \tag{5.5}$$

Then there exists $r_o > 0$, depending only on K, v_o and D, and coordinate charts on the r_o-cylinder

$$C_{r_o} = B_p(r_o) \times [-r_o, r_o],$$

such that the components of the metric $\mathbf{g}_{\alpha\beta}$ are in $C^{1,\alpha} \cap L^{2,p}$, for any $\alpha < 1$, $p < \infty$.

Further, there exists R_o, depending only on K, v_o, D and p, such that, on C_{r_o},

$$||\mathbf{g}_{\alpha\beta}||_{L^{2,p}} \leq R_o. \tag{5.6}$$

Here, the components $\mathbf{g}_{\alpha\beta}$ are the full space-time components of $\mathbf{g}$, and the estimate (5.6) gives bounds on both spatial and time derivatives of $\mathbf{g}$, up to order 2, in L^p, where L^p is measured on spatial slices of C_{r_o}.

This result is formulated in such a way that it is easy to pass to limits. Thus, if one has a sequence of smooth space-times $(\mathbf{M}_i, \mathbf{g}_i)$ satisfying the hypotheses of the Theorem, (with fixed constants K, v_o and uniformly compact domains D), then it follows that, in a subsequence, there is a limit $C^{1,\alpha} \cap L^{2,p}$ space-time $(\mathbf{M}_\infty, \mathbf{g}_\infty)$, defined at least on the r_o-cylinder C_{r_o}. Further, the convergence to the limit is $C^{1,\alpha}$ and weak $L^{2,p}$, and the estimate (5.6) holds on the limit.

We sketch some of the ideas of the proof; full details appear in [9]. First, one constructs a new local time function τ on small cylinders C_{r_1}, with $|\nabla\tau|^2 = -1$, so the flow of $\nabla\tau$ is by time-like geodesics. On the level sets Σ_τ of τ, one constructs spatially harmonic coordinates $\{x_i\}$, (w.r.t. the induced Riemannian metric). This gives a local coordinate system $(\tau, x_1, ..., x_n)$ on small cylinders about p. One then uses the transport or Raychaudhuri equation, together with the Bochner-Weitzenbock formula, (Simons' equation), and elliptic estimates to control $\mathbf{g}_{\alpha\beta}$.

The vacuum Einstein equations are needed in Theorem 5.1 only to prove the 2^{nd} time derivatives $\partial_\tau\partial_\tau g_{0\alpha}$ are in L^p, via use of the Bianchi identity. In place of vacuum space-times, it suffices to have a rather weak bound on the stress-energy tensor in the Einstein equations. All other bounds on $g_{\alpha\beta}$ do not require the Einstein equations.

It seems as if this result should be of use in understanding the structure of the boundary of space-times.

If the volume bound on space-like hypersurfaces in (5.5) is dropped, then it is possible that space-like hypersurfaces may collapse with bounded curvature, as described in §3. Examples of this behavior occur on approach to Cauchy horizons, (as noted in §3 in connection with the Berger collapse and the Taub-NUT metric). More generally, Rendall [38] has proved the following interesting general result: if Σ is a *compact* Cauchy horizon in a smooth vacuum space-time in 3+1 dimensions, then nearby space-like hypersurfaces collapse with bounded curvature on approach to Σ.

B: 2^{nd} **Level Problem.**

While Theorem 5.1 represents a first step, one would like to do much better by replacing the bound on $|\mathbf{R}|_T$ by a bound on the Ricci curvature of $(\mathbf{M}, \mathbf{g})$, or assuming for instance the vacuum Einstein equations. Thus, one may ask if analogues of Theorems 2.2 or 2.3 hold in the Lorentzian setting.

The main ingredients in the proofs of these results are the splitting theorem - a geometric part - and the strong convergence to limits - an analytic part obtained from elliptic estimates for the Ricci curvature. Now one does have a direct analogue of the splitting theorem for vacuum space-times, (or more generally space-times satisfying the time-like convergence condition). Thus, by work of Eschenburg, Galloway and Newman, if $(\mathbf{M}, \mathbf{g})$ is a time-like geodesically complete, (or a globally hyperbolic), vacuum space-time which contains a time-like line, i.e. a complete time-like maximal geodesic, then $(\mathbf{M}, \mathbf{g})$ is flat, cf. [11] and references therein.

In analogy to the Riemannian case, define then the 1-cross $Cro_1(x,T)$ of a Lorentzian 4-manifold $(\mathbf{M}, \mathbf{g})$ at x, in the direction of a unit time-like vector T, to be the length of the longest maximizing geodesic in the direction T, with center point x. For Ω a domain with compact closure in $\mathbf{M}$ and T a smooth unit time-like vector field, define

$$Cro_1(\Omega, T) = \inf_{x \in \Omega} Cro_1(x, T).$$

What is lacking is the regularity boost obtained from elliptic estimates. For space-times, the vacuum equations give a hyperbolic evolution equation, (in harmonic coordinates), for which one does not have a gain in derivatives. However, the smoothness of initial data is preserved under the evolution, until one hits the boundary of the maximal development.

Let $H^s = H^s(U)$ denote the Sobolev space of functions with s weak derivatives in $L^2(U)$, U a compact domain in $\mathbb{R}^3$. For $s > 2.5$, (so that H^s embeds in C^0), and a space-like hypersurface $S \subset (\mathbf{M}, \mathbf{g})$, define the harmonic radius $\rho_s(x)$ of $x \in S$ in the same way as in Definition 2.1, where the components $\mathbf{g}_{\alpha\beta}$ and derivatives are in both space and time directions. For the following, we need only consider $s \in \mathbb{N}^+$, with s large, for instance, $s = 3$.

Now a well-known result of Choquet-Bruhat [17] states that the maximal vacuum H^s development of smooth (C^∞) initial data on S is the same for all s, provided $s > 2.5$. Thus, one does not have different developments of smooth initial data, depending on the degree of desired H^s regularity. Here, one may assume that S is compact, or work locally, within the domain of dependence of S. This qualitative result can be expressed as follows. Let S_t be space-like hypersurfaces obtained by evolution from initial data on

$S = S_0$. If $x_t \in S_t$, then

$$\rho_s(x_t) \geq c_1 \Rightarrow \rho_{s+1}(x_t) \geq c_2, \tag{5.7}$$

where c_2 depends on c_1 and the (C^∞) initial data on S_0.

We raise the following problem of whether the qualitative statement (5.7) can be improved to a *quantitative* statement.

Regularity Problem. Can the estimate (5.7) be improved to an estimate

$$\inf_{x_t \in S_t} \rho_{s+1}(x_t) \geq c_0 \inf_{x_t \in S_t} \rho_s(x_t), \tag{5.8}$$

where c_0 depends only on the initial data on S? One may assume, w.l.o.g, that $t \leq 1$.

The important point of (5.8) over (5.7) is that the estimate (5.8) is scale-invariant. Here, we recall that $\rho_s(x)$ measures the degree of concentration of derivatives of the metric in H^s, so that $\rho_s \to 0$ corresponds to blow-up of the metric in H^s locally.

If (5.8) holds, it serves as an analogue of the regularity boost. In such circumstances, one can imitate the proof of Theorems 2.2 or 2.3 to obtain similar results for sequences of space-times $(\mathbf{M}, \mathbf{g}_i)$.

In fact, the validity of (5.8) would have numerous interesting applications, even if it could be established under some further restrictions or assumptions.

Suppose next one drops any assumption on the 1-cross of $(\mathbf{M}, \mathbf{g})$ and maintains only a lower bound on the volumes of geodesic balls, as in (5.5), on space-like hypersurfaces. This leads directly to issues of singularity formation and the structure of the boundary of the vacuum space-time, where comparatively little is known mathematically.

A useful problem, certainly simple to state, is the following: for simplicity, we work in the context of compact, (i.e. closed, without boundary), Cauchy surfaces.

Sandwich Problem.

Let $(\mathbf{M}, \mathbf{g}_i)$ be a sequence of vacuum space-times, and let Σ_i^1, Σ_i^2 be two compact Cauchy surfaces in $\mathbf{M}$, with Σ_i^2 to the future of Σ_i^1 and with

$$1 \leq dist_{\mathbf{g}}(x, \Sigma_i^1) \leq 10,$$

for all $x \in \Sigma_i^2$. Suppose the Cauchy data (g_i^j, K_i^j), $j = 1, 2$ on each Cauchy surface are uniformly bounded in H^s for some fixed $s > 2.5$, possibly large. Hence the data (g_i^j, K_i^j) converge, in a subsequence and weakly in H^s, to limit H^s Cauchy data g_∞^j, K_∞^j on Σ^j.

Do the vacuum space-times $A_i(1,2) \subset (M, g_i)$ between Σ^1 and Σ^2 converge, weakly in H^s, to a limit space time,

$$(A_i(1,2), g_i) \to (A_\infty, g_\infty)? \tag{5.9}$$

This question basically asks if a singularity can form between Σ_i^1 and Σ_i^2 in the limit. It is unknown even if there could be only a single singularity at an isolated point (event) $x_0 \in (A_\infty, g_\infty)$.

The existence of such a singularity may be related to the Choptuik solution. However, both the existence and the smoothness properties of the Choptuik solution have not been established well mathematically; cf. [30] for an interesting discussion.

Such a limit singularity would be naked in a strange way. It could be detected on Σ^2, since light rays from it propagate to Σ^2, but on Σ^2, no remnant of the singularity is detectable, since the data is smooth on Σ^2. Thus, the singularity is invisible to the future (or past) in a natural sense.

A resolution of this problem would be useful in understanding, for instance, limits of the asymptotically simple vacuum perturbations of deSitter space, given by Friedrich's theorem [24]. The sandwich problem above asks: suppose one has control on the space-time near past and future space-like infinity $\mathcal{I}^\pm$, does it follow that one has control in between?

Similar questions can be posed for non-compact Cauchy surfaces, and relate for instance to limits of the AF perturbations of Minkowski space given by Christodoulou-Klainerman, [19].

6. Future Asymptotics and Geometrization of 3-Manifolds.

In this section, we give some applications to the future asymptotic behavior of cosmological spaces times.

Let $(\mathbf{M}, \mathbf{g})$ be a vacuum cosmological space-time, i.e. $(\mathbf{M}, \mathbf{g})$ contains a compact Cauchy surface Σ of constant mean curvature (CMC). We assume throughout this section that Σ is of non-positive Yamabe type, so that Σ admits no metric of positive scalar curvature. It is well-known that Σ then embeds in a foliation $\mathcal{F}$ by CMC Cauchy surfaces Σ_τ, all diffeomorphic to $\Sigma = \Sigma_1$, and parametrized by their mean curvature τ. The parameter τ thus serves as a time function, taking values in

$$\tau \in (-\infty, 0), \tag{6.1}$$

with τ increasing towards the future in $(\mathbf{M}, \mathbf{g})$. The sign of the mean curvature is chosen so that $vol_{g_\tau}\Sigma_\tau$ is increasing with increasing τ, i.e. expanding towards the future. The foliated region $\mathbf{M}_\mathcal{F}$ is thus a subset of $\mathbf{M}$, although in general $\mathbf{M} \neq \mathbf{M}_\mathcal{F}$.

Suppose that $(\mathbf{M}, \mathbf{g})$ is geodesically complete to the future of Σ, and that the future is foliated by CMC Cauchy surfaces, i.e. $\mathbf{M} = \mathbf{M}_\mathcal{F}$ to the future of Σ. These are of course strong assumptions, but are necessary if one wants to understand the future asymptotic behavior of $(\mathbf{M}, \mathbf{g})$ without the complicating issue of singularities.

The topology of Σ_τ is fixed, and so the metrics g_τ induced on Σ_τ by the ambient metric $\mathbf{g}$ give rise to a curve of Riemannian metrics on the fixed manifold Σ. It is not hard to see that $vol_{g_\tau}\Sigma \to \infty$ as $\tau \to 0$, and typically, the metrics g_τ become flat, due to the expansion, (compare with the discussion in §1).

This is of course not very interesting. As in §1 and §4, to study the asymptotic behavior, one should rescale by the distance to a fixed base point or space-like hypersurface. In this case, the distance is the time-like Lorentzian distance. Thus, for x to the future of $\Sigma = \Sigma_{-1}$, let $t(x) = dist_{\mathbf{g}}(x, \Sigma)$ and let

$$t_\tau = t_{max}(\tau) = max\{t(x) : x \in \Sigma_\tau\} = dist_{\mathbf{g}}(\Sigma_\tau, \Sigma). \tag{6.2}$$

It is natural to study the asymptotic behavior of the metrics

$$\bar{g}_\tau = t_\tau^{-2} g_\tau, \tag{6.3}$$

on Σ_τ. Observe that in the rescaled space-time (M, $\bar{\mathbf{g}}_\tau$), the distance of $(\Sigma_\tau, \bar{g}_\tau)$ to the "initial" singularity, (big bang), tends towards 1, as $\tau \to 0$. Any other essentially distinct scaling would have the property that the distance to the initial singularity tends towards 0 or ∞, and so is not particularly natural.

We need the following definition.

Definition 6.1. Let Σ be a closed, oriented, connected 3-manifold, of non-positive Yamabe type. A *weak* geometrization of Σ is a decomposition

$$\Sigma = H \cup G, \tag{6.4}$$

where H is a finite collection of complete, connected hyperbolic manifolds, of finite volume, embedded in Σ, and G is a finite collection of connected graph manifolds, embedded in Σ. The union is along a finite collection of embedded tori $\mathcal{T} = \cup T_i = \partial H = \partial G$.

A *strong* geometrization of Σ is a weak geometrization as above, for which each torus $T_i \in \mathcal{T}$ is incompressible in Σ, i.e. the inclusion of T_i into Σ induces an injection of fundamental groups.

Of course it is possible that the collection $\mathcal{T}$ of tori dividing H and G is empty, in which case weak and strong geometrizations coincide. In such a situation, Σ is then either a closed hyperbolic manifold or a closed graph manifold. For a strong geometrization, the decomposition (6.4) is unique up to isotopy, but this is certainly not the case for a weak geometrization, c.f. the end of §3.

In general, no fixed metric g on Σ will realize the decomposition (6.4), unless $\mathcal{T} = \emptyset$. This is because the complete hyperbolic metric on H does not extend to a metric on Σ. However, one can find sequences of metrics g_i on Σ which limit on a geometrization of Σ in the sense of (6.4). Thus, the metrics g_i may be chosen to converge to the hyperbolic metric on larger and larger compact subsets of H, to be more and more collapsed with bounded curvature on G, and such that their behavior matches far down the collapsing hyperbolic cusps.

Next, to proceed further, we need to impose a rather strong curvature assumption on the ambient space-time curvature. Thus, suppose there is a constant $C < \infty$ such that, for x to the future of Σ,

$$|\mathbf{R}|(x) + t(x)|\nabla\mathbf{R}|(x) \leq C \cdot t^{-2}(x). \tag{6.5}$$

Here, the curvature norm $|\mathbf{R}|$ may be given by $|\mathbf{R}|_T$ as in (5.3), where T is the unit normal to the foliation Σ_τ. Since $(\mathbf{M}, \mathbf{g})$ is vacuum, this is equivalent to $|\mathbf{R}|^2 = |E|^2 + |B|^2$, where E, B is the electric/magnetic decomposition of $\mathbf{R}$, $E(X,Y) =< \mathbf{R}(X,T)T, Y >$, $B(X,Y) =< (*\mathbf{R})(X,T)T, Y >$ with X, Y tangent to the leaves. Similarly, $|\nabla\mathbf{R}|^2 = |\nabla E|^2 + |\nabla B|^2$.

The bound (6.5) is scale invariant, and analogous to the bound (4.9) or (4.10) for stationary space-times, (where it of course holds in general). The bound on $|\nabla\mathbf{R}|$ in (6.5) is needed only for technical reasons, (related to Cauchy stability), and may be removed in certain natural situations.

It is essentially obvious, and in any case easily verified, that the curvature assumption (6.5) holds for the class of expanding Bianchi cosmological models, (where the leaves Σ_τ are locally homogeneous). It is natural to conjecture that it also holds at least for perturbations of the Bianchi models. Similarly, we conjecture it holds for all vacuum Gowdy space-times.

In fact, there are no known cosmological space-times $(\mathbf{M}, \mathbf{g})$, geodesically complete to the future, for which (6.5) is known to fail.

The discussion above leads to the following result from [8], to which we refer for further discussion and details.

Theorem 6.2. *Let $(\mathbf{M}, \mathbf{g})$ be a cosmological space-time of non-positive Yamabe type. Suppose that the curvature assumption (6.5) holds, and that $M_{\mathcal{F}} = \mathbf{M}$ to the future of Σ.*

Then $(\mathbf{M}, \mathbf{g})$ is future geodesically complete and, for any sequence $\tau_i \to 0$, the slices $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$, cf. (6.3), have a subsequence converging to a weak geometrization of Σ, in the sense following Definition 6.1.

We indicate some of the basic ideas in the proof. The first step is to show that the bound (6.5) on the ambient curvature $\mathbf{R}$, in this rescaling, gives uniform bounds on the intrinsic and extrinsic curvature of the leaves Σ_τ. The proof of this is similar to the proof of Theorem 5.1.

Given this, one can then apply the Cheeger-Gromov theory, as described in §2-3. Given any sequence $\tau_i \to 0$, there exist subsequences which either converge, collapse or form cusps. From the work in §3, one knows that the regions of $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$ which (fully) collapse, or which are sufficiently collapsed, are graph manifolds. This gives rise to the region G in (6.4). It remains to show that, for any fixed $\varepsilon > 0$, the ε-thick region Σ^ε of $(\Sigma_{\tau_i}, \bar{g}_{\tau_i})$ converges to a hyperbolic metric.

The main ingredient in this is the following volume monotonicity result:

$$\frac{vol_{g_\tau}\Sigma_\tau}{t_\tau^3} \downarrow, \tag{6.6}$$

i.e. the ratio is monotone non-increasing in the distance t_τ. This result is analogous to the Fischer-Moncrief monotonicity of the reduced Hamiltonian along the CMC Einstein flow, cf. [23]. The monotonicity (6.6) is easy to prove, and is an analogue of the Bishop-Gromov volume monotonicity (2.12). It follows from an analysis of the Raychaudhuri equation, much as in the Penrose-Hawking singularity theorems, together with a standard maximum principle.

Moreover, the ratio in (6.6) is constant on some interval $[\tau_1, \tau_2]$ if and only if the annular region $\tau^{-1}(\tau_1, \tau_2)$ is a time annulus in a flat Lorentzian cone

$$\mathbf{g_o} = -dt^2 + t^2 g_{-1}, \tag{6.7}$$

where g_{-1} is a hyperbolic metric. Again, the ratio in (6.6) is scale invariant, and so

$$\frac{vol_{g_\tau}\Sigma_\tau}{t_\tau^3} = vol_{\bar{g}_\tau}\Sigma_\tau. \tag{6.8}$$

In the non-collapse situation, $vol_{\bar{g}_\tau}\Sigma_\tau$ is uniformly bounded away from 0 as $\tau \to 0$, (i.e. $t_\tau \to \infty$), and hence converges to a non-zero limit. On approach to the $\tau = 0$ limit, the ratio (6.6) tends to a constant, and hence the corresponding limit manifolds are of the form (6.7). This implies that ε-thick regions converge to hyperbolic metrics, giving rise to the H factor in (6.4).

References

[1] M. Anderson, Convergence and rigidity of manifolds under Ricci curvature bounds, Inventiones Math., **106**, (1990), 429-445.

[2] M. Anderson, Short geodesics and gravitational instantons, Jour. Diff. Geom., **31**, (1990), 265-275.

[3] M. Anderson and J. Cheeger, C^α compactness for manifolds with Ricci curvature and injectivity radius bounded below, Jour. Diff. Geom., **35**, (1992), 265-281.

[4] M. Anderson, Hausdorff perturbations of Ricci-flat metrics and the splitting theorem, Duke Math. Jour., **68**, (1992), 67-82.

[5] M. Anderson, Extrema of curvature functionals on the space of metrics on 3-manifolds, Calc. Var. & P.D.E., **5**, (1997), 199-269.

[6] M. Anderson, On stationary vacuum solutions to the Einstein equations, Ann. Henri Poincaré, **1**, (2000), 977-994.

[7] M. Anderson, On the structure of solutions to the static vacuum Einstein equations, Ann. Henri Poincaré, **1**, (2000), 995-1042.

[8] M. Anderson, On long-time evolution in general relativity and geometization of 3-manifolds, Comm. Math. Phys., **222**, (2001), 533-567.

[9] M. Anderson, Regularity for Lorentz metrics under curvature bounds, (to appear).

[10] L. Andersson and V. Moncrief, (to appear).

[11] J. Beem, P. Ehrlich and K. Easley, Global Lorentzian Geometry, 2^{nd} Edition, Marcel Dekker, New York, (1996).

[12] R. Beig and W. Simon, Proof of a multipole conjecture due to Geroch, Comm. Math. Phys., **78**, (1980), 75-82.

[13] J. Bičák, Selected solutions of Einstein's field equations: their role in General Relativity and Astrophysics, in Einstein's Field Equations and their Physical

Applications, Lecture Notes in Physics, **540**, Springer Verlag, Berlin, (2000), 1-126.

[14] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. Jour. Math., **92**, (1970), 61-75.

[15] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of non-negative Ricci curvature, Jour. Diff. Geom., **6**, (1971), 119-128.

[16] J. Cheeger and M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded, I, II, Jour. Diff. Geom., **23**, (1986), 309-346 and **32**, (1990), 269-298.

[17] Y. Choquet-Bruhat, Solutions C^{∞} d'equations hyperboliques non-lineares, C. R. Acad. Sci. Paris, Ser A-B, **272**, (1971), A386-A388.

[18] Y. Choquet-Bruhat and V. Moncrief, Future global in time Einsteinian spacetimes with $U(1)$ isometry group, Ann. Henri Poincare, **2**, (2001), 1007-1064.

[19] D. Christodoulou and S. Klainerman, The Global Nonlinear Stability of the Minkowski Space, Princeton Univ. Press, Princeton, (1993).

[20] C.J.S. Clarke, The Analysis of Space-Time Singularities, Camb. Lect. Notes in Physics, **1**, Camb. Univ. Press, (1993).

[21] D. Deturck and J. Kazdan, Some regularity theorems in Riemannian geometry, Ann. Sci. Ecole Norm. Sup, **14**, (1981), 249-260.

[22] A. Einstein and W. Pauli, Non-existence of regular stationary solutions of relativistic field equations, Annals of Math., ser. 2, **44**, (1943), 131-137.

[23] A. Fischer and V. Moncrief, The reduced Hamiltonian of general relativity and the σ-constant of conformal geometry, in Math. and Quantum Aspects of Relativity and Cosmology, S. Cotsakis and G.W. Gibbons, (Eds.), Springer Lecture Notes in Physics, **537**, (2000), 70-101.

[24] H. Friedrich, On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure, Comm. Math. Phys., 107, (1986), 587-609.

[25] D. Garfinkle and M. Melvin, Generalized magnetic universe solutions, Phys. Rev. **D50**, (1994), 3859-3866.

[26] G.W. Gibbons and S.W. Hawking, Classification of gravitational instanton symmetries, Comm. Math. Phys., **66**, (1979), 291-310.

[27] M. Gromov, Structures Metriques Pour Les Varieties Riemanniennes, Cedic-Nathan, Paris, (1981).

[28] M. Gromov, Almost flat manifolds, Jour. Diff. Geom., **13**, (1978), 231-241.

[29] M. Gromov, Volume and bounded cohomology, Publ. Math. I.H.E.S., **56**, (1982), 5-99.

[30] C. Gundlach, Critical phenomena in gravitational collapse, Living Reviews in Relativity, **2:4**, (1999), www.livingreviews.org

[31] S. Hawking and G. Ellis, The Large Scale Structure of Space-Time, Cambridge Univ. Press, London, (1973).

[32] J. Jost and H. Karcher, Geometrische Methoden zur Gewinnung von a-priori Schranken für harmonische Abbildungen, Manuscripta Math., **40**, (1982), 21-71.

[33] A. Lichnerowicz, Theories Relativistes de la Gravitation et de L'Electromagnetisme, Masson, Paris, (1955).

[34] M. Melvin, Pure magnetic and electric geons, Physics Letters, **8:1**, (1964), 65-68.

[35] R. Myers, Higher dimensional black holes in compactified space-times, Phys. Rev. D., **35**, (1987), 455-466.

[36] P. Orlik, Seifert Manifolds, Springer Lecture Notes in Math., **291**, Springer Verlag, New York, (1972).

[37] R. Penrose, Any space-time has a plane-wave as a limit, in Differential Geometry and Relativity, Ed. Cahan and Flato, (1976), 271-275.

[38] A. Rendall, Compact null hypersurfaces and collapsing Riemannian manifolds, Math. Nachrichten, **193**, (1998), 111-118.

[39] W. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc, **6**, (1982), 357-381.

[40] F. Tipler, C. Clarke and G. Ellis, Singularities and horizons: a review article, in Gen. Rel. and Gravitation, vol. 2, A. Held, (Ed.), Plenum Press, New York, (1980), 87-206.

[41] F. Waldhausen, Eine klasse von 3-dimensionalen Mannifaltigkeiten, I, II, Invent. Math. **3**, (1967), 308-333 and **4**, (1967), 87-117.

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